

GENERAL AFFINE ADJUNCTIONS, NULLSTELLENSÄTZE, AND DUALITIES

OLIVIA CAMELLO, VINCENZO MARRA, AND LUCA SPADA

ABSTRACT. We develop an abstract categorical framework that generalises the classical “system-solution” adjunction in algebraic geometry, proving that such adjunctions take place in a multitude of contexts. We then look further into these geometric adjunctions at different levels of generality, from syntactic categories to equational classes of algebras. In doing so, we discuss the relationships between the dualities induced by our framework and the theory of dualities generated by a “schizophrenic” object. Notably, classical dualities like Stone duality for Boolean algebras, Gelfand duality for commutative C^* -algebras, Pontryagin duality for Abelian groups, turn out to be special instances of this framework. To determine how such general adjunctions restrict to dualities we prove abstract analogues of Hilbert’s Nullstellensatz and Gelfand-Kolmogorov-Stone lemma, completely characterising the fixed points on one side of the adjunction.

CONTENTS

Part 1. Prologue	2
1. Introduction	2
1.1. The classical affine adjunction	2
1.2. The universal algebra framework	3
1.3. The general affine adjunction	5
1.4. Applications to duality theory	6
1.5. Related literature	6
Part 2. The general adjunction	7
2. The weak Nullstellensatz, and the weak affine adjunction	7
2.1. The category \mathbf{D} of subobjects and definable morphisms	7
2.2. The category \mathbf{R} of relations and relation-preserving morphisms	7
2.3. The Galois connection (\mathbf{C}, \mathbf{V})	8
2.4. The weak <i>Nullstellensatz</i>	8
2.5. The functor $\mathcal{C}: \mathbf{D} \rightarrow \mathbf{R}$	9
2.6. The functor $\mathcal{V}: \mathbf{R} \rightarrow \mathbf{D}$	10
2.7. The weak adjunction	10
3. The general adjunction	10
4. Further general theory	12
4.1. Comprehensiveness of the framework	12
4.2. The case of a representable $\mathcal{J}: \mathbf{T} \rightarrow \mathbf{S}$	14
4.3. The setting of syntactic categories	18

4.4. Recovering Diers' "system-solution" adjunction	20
Part 3. The specialisation to varieties of algebras	21
5. The general setting	21
6. The algebraic affine adjunction	24
6.1. The quotient D^q : Affine subsets	27
6.2. The quotient R^q : Presented algebras	27
6.3. Algebraic affine adjunction	28
7. The algebraic <i>Nullstellensatz</i>	28
7.1. A Stone-Gelfand-Kolmogorov Lemma	28
7.2. Transforms	30
7.3. The algebraic <i>Nullstellensatz</i>	31
8. The topological <i>Nullstellensatz</i>	33
Part 4. Three classical examples and one epilogue	35
9. The classical affine adjunction	35
10. Stone duality for Boolean algebras	36
11. Gelfand duality for C^* -algebras	37
12. Conclusions	39
Acknowledgements.	40
References	41

Part 1. Prologue

1. INTRODUCTION

We are concerned in this paper with a generalisation of the affine adjunction of classical algebraic geometry to algebraic and categorical settings, and with the resulting relationship to the theory of dualities. We establish a general framework that encompasses several important categorical dualities in mathematics, as well as several attempts to unify them. The paper thus spans a number of different topics, often at varying levels of generality. We offer an introduction that is perhaps longer than customary, in the hope of providing motivation and guidance for the interested reader.

1.1. The classical affine adjunction. In classical affine algebraic geometry, one studies solutions to systems of polynomial equations with coefficients in an algebraically closed field k . For any subset R of the polynomial ring over finitely many variables $k[X] := k[X_1, \dots, X_n]$, there remains defined the (possibly infinite) system of equations:

$$p(X_1, \dots, X_n) = 0, \quad p \in R. \quad (1)$$

Let us write $\mathbb{V}(R) \subseteq k^n$ for the set of solutions of (1) over k^n , where k^n is the affine n -space over k . Then $\mathbb{V}(R)$ is the *affine set defined by R* . Since $k[X]$ is Noetherian by Hilbert's Basis Theorem, it is no loss of generality to assume that R be finite.

Conversely, for any subset $S \subseteq k^n$ we can consider the set $\mathbb{C}(S) \subseteq k[X]$ of polynomials that vanish over S , which is automatically an ideal. Then $\mathbb{C}(S)$ is the *ideal defined by S* . Again by Hilbert's Basis Theorem, the quotient k -algebra $k[X]/\mathbb{C}(S)$ —the *co-ordinate ring* of the affine set S — is *finitely presentable*.

Writing 2^E for the power set of the set E , we obtain functions (implicitly indexed by n)

$$\mathbb{C}: 2^{k^n} \longrightarrow 2^{k[X]}, \quad (2)$$

$$\mathbb{V}: 2^{k[X]} \longrightarrow 2^{k^n} \quad (3)$$

that yield a (contravariant) Galois connection. The fixed points of the closure operator $\mathbb{V} \circ \mathbb{C}$ are then precisely the affine sets in k^n . Since $\mathbb{V} \circ \mathbb{C}$ is a *topological* closure operator —i.e. it commutes with finite unions— affine algebraic sets are the closed sets of a topology on k^n , namely, the *Zariski topology*. The fixed points of the dual closure operator $\mathbb{C} \circ \mathbb{V}$, on the other hand, may be identified thanks to Hilbert’s *Nullstellensatz*: they are precisely the *radical ideals* of $k[X]$, that is, those ideals that coincide with the intersection of all prime ideals containing them. The *Nullstellensatz* thus characterises co-ordinate rings, for $k[X]/I$ is one such if, and only if, I is radical. Since radical ideals may in turn be elementarily characterised as those ideals I such that $k[X]/I$ has no non-zero nilpotents, co-ordinate rings are precisely the finitely presented nilpotent-free (or *reduced*) k -algebras.

The Galois connection given by the pair (\mathbb{C}, \mathbb{V}) in (2–3) can be made functorial. On the algebraic side we consider the category of finitely presented k -algebras with their homomorphism. On the geometric side we take as objects subsets of k^n , for each finite n , by which we mean sets S equipped with a specific embedding $S \hookrightarrow k^n$. It is important not to blur the distinction between S itself — a mere set— and $S \hookrightarrow k^n$ —an object of our category. Indeed, arrows in the geometric category are to be defined affinely, i.e. by restriction from k^n . An arrow from $S \hookrightarrow k^n$ to $T \hookrightarrow k^m$ is a *regular map* $S \rightarrow T$, that is, the equivalence class of a *polynomial function* $f: k^n \rightarrow k^m$ such that f throws S onto T ; two such functions are equivalent if, and only if, they agree on S . There is a functor that associates to each regular map $S \rightarrow T$ a contravariant homomorphism of the (automatically presented) co-ordinate rings of $\mathbb{V} \circ \mathbb{C}(T)$ and $\mathbb{V} \circ \mathbb{C}(S)$. And there is a companion functor that associates to each homomorphism of presented k -algebras $k[X]/I \rightarrow k[Y]/J$, with $Y = \{Y_1, \dots, Y_m\}$ and J an ideal of $k[Y]$, a contravariant regular map $\mathbb{V}(J) \rightarrow \mathbb{V}(I)$. The two functors yield a contravariant adjunction; upon restricting each functor to the fixed points in each domain, one obtains the classical duality (=contravariant equivalence) between affine algebraic varieties and their co-ordinate rings. Compare¹ e.g. [24, Corollary 3.8].

1.2. The universal algebra framework. A first aim of this paper is to generalise the classical affine adjunction above to any *variety of algebras*, whether finitary to infinitary. We assume some familiarity with Birkhoff’s theory of general, or “universal”, algebra; for background see e.g. [8, 16, 26, 10]. Henceforth, variety (of algebras) means “possibly infinitary variety (of algebras)” in the sense of Słominsky [44] and Linton [31] (after Lawvere [30]). The main observation is that in any variety, the *free algebras* play the same rôle as the ring of polynomials in the above correspondence. Ideals of the ring of polynomials become then, in full generality, congruences of some free algebra, while the ground field k can be substituted for any algebra A in the variety. We refer the reader to Table 1 below, for a schematic

¹Terminology: Hartshorne’s corollary is stated for irreducible varieties, which he calls varieties *tout court*.

translation of the main concepts in the adjunctions. In Part 3 we show that the classical affine adjunction extends *verbatim* to this general algebraic setting.

It is important to note that in the geometric adjunction, co-ordinate rings are *presented*, that is, they are not merely isomorphic to a ring of the form $k[X]/I$: they come with a specific defining ideal I . By an easy general argument relying on the Axiom of Choice —cf. Remark 6.10 below— the category of finitely presented k -algebras is equivalent to that of finitely presentable k -algebras (morphisms being the ring homomorphisms in each case), whether actually presented or not. Nonetheless, for our purposes here the presented and the *presentable* objects are to be kept distinct. We shall indicate by \mathbf{V}_p the category of presented algebras in the variety \mathbf{V} .

Algebraic geometry	Universal algebra	Categories
Ground field k	Any algebra A in \mathbf{V}	Functor $\mathcal{J} : \mathbf{T} \rightarrow \mathbf{S}$
Class of k -algebras	Any variety \mathbf{V}	Category \mathbf{R}
$k[X_1, \dots, X_n]$	Free algebras	Objects in \mathbf{T}
Ideals	Congruences	Subsets of $\text{hom}_{\mathbf{T}}^2(t, \Delta)$ with t in \mathbf{T}
Assignment $k[X_1, \dots, X_n] \rightarrow k$	Assignment $\mathcal{F}(\mu) \rightarrow A$	Object Δ in \mathbf{T}
Regular map	Definable map	Restriction of $\mathcal{J}(f)$
Co-ordinate algebra of S	Algebra presented by $\mathbb{C}(S)$	Pair $(t, \mathbb{C}(S))$ in \mathbf{R}
Affine variety	$\mathbb{V} \circ \mathbb{C}$ -closed set	Pair $(t, \mathbb{V}(R))$ in \mathbf{S}

TABLE 1. Corresponding concepts in the geometric, algebraic, and categorical setting.

In Corollary 6.13 we obtain the adjunction between \mathbf{V}_p^{op} , the opposite of the category of presented \mathbf{V} -algebras, and the category of subsets of (the underlying set of) A^μ , as μ ranges over all cardinals, with definable maps as morphisms. The functors that implement the adjunction act on objects by taking a subset $R \subseteq \mathcal{F}(\mu) \times \mathcal{F}(\mu)$ —that is, a “system of equations in the language of \mathbf{V} ”— to its solution set $\mathbb{V}(R) \subseteq A^\mu$, where $\mathbb{V}(R)$ is the set of elements of A^μ such that each pair of terms in R evaluate identically over it; and a subset $S \hookrightarrow A^\mu$ to its “co-ordinate \mathbf{V} -algebra”, namely, $\mathcal{F}(\mu)/\mathbb{C}(S)$, where $\mathbb{C}(S)$ is the congruence on $\mathcal{F}(\mu)$ consisting of all pairs of terms that evaluate identically at each element of S . Please see section 6 for details.

To identify the fixed points of this general affine adjunction on the algebraic side, we prove an appropriate generalisation of the *Nullstellensatz*. The final result is stated as Theorem 7.7. The identification of an appropriate notion of radical congruence (in part (ii) of the theorem) leads to a result formally analogous to the ring-theoretic *Nullstellensatz*. Additionally, the identification of an appropriate type of representation for those \mathbf{V} -algebras that are fixed under the adjunction (in part (iii) of the theorem) leads to a result reminiscent of Birkhoff’s Subdirect Representation Theorem. Failure of the latter for infinitary varieties is irrelevant for our purposes here. In fact, while our Theorem 7.7 may be conceived of as a version of the Subdirect Representation theorem that is “relative to the ground

algebra A ", it is formally incomparable to Birkhoff's result: neither statement entails the other, in general. See section 5 for details and further comments.

To characterise the fixed points on the *affine* side, we use the fact that in several cases the composition $\mathbb{V} \circ \mathbb{C}$ gives a topological closure operator. The topology induced by $\mathbb{V} \circ \mathbb{C}$ is readily seen to be a generalisation of the Zariski topology (see e.g., [24, Chapter 1]). We therefore provide some sufficient condition to characterise the fixed points on this side as topological A -compact sets ([46]).

1.3. The general affine adjunction. The general affine adjunction of Corollary 6.13 can be lifted from the algebraic setting to a more general categorical context. This we do in Part 2 of the paper, thus achieving our second aim. Conceptually, the key ingredient in the algebraic construction sketched above is the functor $\mathcal{I}_A: \mathbf{T} \rightarrow \mathbf{Set}$. In the categorical abstraction, the basic *datum* is any functor $\mathcal{I}: \mathbf{T} \rightarrow \mathbf{S}$, which can be conceived as the *interpretation* of the "syntax" \mathbf{T} into the "semantics" \mathbf{S} , along with a distinguished object Δ of \mathbf{T} . (In the algebraic specialisation, Δ is $\mathcal{F}(1)$, the free singly generated V -algebra.) Here \mathbf{T} and \mathbf{S} are simply arbitrary (locally small²) categories. Out of these data, we construct two categories \mathbf{D} and \mathbf{R} of subobjects and relations, respectively.

The category \mathbf{D} abstracts that of sets affinely embedded into A^μ ; here, sets are replaced by objects of \mathbf{S} , the powers A^μ are replaced by objects $\mathcal{I}(t)$ as t ranges over objects of \mathbf{T} , and the morphisms of \mathbf{S} that are "definable" are declared to be those in the range of \mathcal{I} . The category \mathbf{R} abstracts the category of relations (not necessarily congruences) on the free V -algebras $\mathcal{F}(\mu)$; that is, its objects are relations on the hom-set $\text{hom}_{\mathbf{T}}(t, \Delta)$, as t ranges over objects of \mathbf{T} . Arrows are \mathbf{T} -arrows that preserve the given relations.

It is possible, in this setting, to define the operator \mathbb{C} in full generality. In order to define an appropriate abstraction of the operator \mathbb{V} , we need to require that \mathbf{S} has enough limits (Assumption 1 below), because "solutions" to "systems of equations" are computed by intersecting solutions to "single equations". The pair (\mathbb{C}, \mathbb{V}) yields a Galois connection (Lemma 2.4) that satisfies an appropriate abstraction of the *Nullstellensatz*, as we show in Theorem 2.5. Moreover, the Galois connection functorially lifts to an adjunction between \mathbf{D} and \mathbf{R} ; see Theorem 2.8. This is to be considered a weak form of the algebraic adjunction, because in the algebraic setting one can additionally take quotients of the categories \mathbf{D} and \mathbf{R} that have semantic import.

One would like to identify pairs of definable morphisms in \mathbf{D} , if they agree on the given "affine subobject". Similarly, one would like to identify morphisms that agree on the same "presented object", in the appropriate abstract sense. See section 2 for details. This can be done via appropriate equivalence relations that lead us to quotient categories \mathbf{D}^q and \mathbf{R}^q . However, in order for the adjunction between \mathbf{D} and \mathbf{R} to descend to the quotients, it is necessary to impose a condition on the object Δ . More precisely, we find that it suffices to assume that Δ be an \mathcal{I} -coseparator; please see Definition 3.5 and Lemma 3.6. (In the algebraic specialisation, we prove that this assumption on $\Delta = \mathcal{F}(1)$ is automatically satisfied; see Lemma 6.11.) Under this additional assumption (Assumption 2 below) we obtain our general affine adjunction between \mathbf{D}^q and \mathbf{R}^q , Theorem 3.8. In section 4 of the paper we

²All categories in this paper are assumed to be locally small.

develop some further theory with an eye towards comparing our results to the existing literature.

1.4. Applications to duality theory. Our third and final aim in this paper is to illustrate the connection between the theory of dualities, and the general affine adjunctions that we have hitherto summarised. This we do in Part 3, where we select three landmark duality theorems in order to illustrate different aspects of our construction. Some familiarity with duality theory is assumed here. By way of a preliminary, in section 9 we show in detail how the classical affine adjunction can be recovered as a rather direct special case of the algebraic affine adjunction. In section 10 we frame Stone duality for Boolean algebras in our setting. This provides the most classical example of a duality for a finitary variety of algebras. In section 11 we do the same for Gelfand duality between commutative unital C^* -algebras and compact Hausdorff spaces, an important example of a duality for an infinitary variety of algebras. Our treatment of Gelfand duality stresses its analogy with Stone duality for Boolean algebras.

1.5. Related literature. Before turning to the proofs, we comment on related literature. The idea of generalising the classical affine adjunction to an algebraic setting is far from new. In [18] and subsequent papers, various elements of abstract algebraic geometry for finitary varieties of algebras are developed. The authors use this to apply geometric methods in universal algebra. However, they do not relate their theory to duality theory, so the overlap with our results is modest.

In [19] Diers develops a framework generalising the classical affine adjunction for rings of polynomials to the context of (possibly infinitary) algebraic theories. He establishes, for any algebra L in a given variety, an adjunction between a category of “affine subsets” over L and a category of “algebraic systems”, as well as an adjunction between a category of “affine algebraic sets” and the category of algebras of the given sort, specialising to a duality between the former category and a category of “functional algebras” over L . The notion of (algebraic) affine set in the sense of Diers is significantly different from ours: indeed, it amounts to a pair $(X, A(X))$ consisting of a set X and of a subalgebra of the algebra L^X . Nonetheless, in section 4.4 we show that Diers’ “system-solution” adjunction can be obtained from our general categorical framework with an appropriate choice of the parameters. Another important difference between Diers’ approach and ours consists in the fact that the categories of algebras involved in his other adjunction are not categories of *presented algebras* (i.e., algebras equipped with a presentation) as it is the case in our setting, nor the objects of his category of affine algebraic sets are presented as *subsets* of affine spaces.

There is also a strong connection between our approach and the theory of dualities generated by a dualising object (see e.g. [5], [15], and [42]). In section 4.2 we show that, whenever \mathbf{S} is the category of sets and maps, and \mathcal{S} is representable, our adjunction can be seen as one induced by a dualising object. Moreover, in section 4.1 we show that every duality between categories satisfying mild requirements can be obtained from the general categorical framework developed in Part 2.

Finally, we mention that the connection between a general *Nullstellensatz* theorem for varieties of algebras and Birkhoff’s subdirect representation theorem is addressed in [45], although in a context different from ours.

Part 2. The general adjunction

2. THE WEAK NULLSTELLENSATZ, AND THE WEAK AFFINE ADJUNCTION

If x and y are objects in a category \mathbf{C} , and $f: x \rightarrow y$ is an arrow in \mathbf{C} , we write $\text{hom}_{\mathbf{C}}(x, y)$ to denote the collection of arrows in \mathbf{C} from x to y , and $\text{dom } f$ to denote the domain x of f . We consider the following.

- Two categories \mathbf{T} and \mathbf{S} .
- A functor $\mathcal{J}: \mathbf{T} \rightarrow \mathbf{S}$.
- An object Δ of \mathbf{T} .

2.1. The category \mathbf{D} of subobjects and definable morphisms. Objects are all pairs (t, s) where t is \mathbf{T} -object and $s: \text{dom } s \rightarrow \mathcal{J}(t)$ is an \mathbf{S} -subobject. Arrows $(t, s) \rightarrow (t', s')$ are the \mathbf{T} -arrows $f: t \rightarrow t'$ such that $\mathcal{J}(f) \circ s$ factors through s' ; that is, there exists an \mathbf{S} -arrow $g: \text{dom } s \rightarrow \text{dom } s'$ such that the diagram

$$\begin{array}{ccc} \mathcal{J}(t) & \xrightarrow{\mathcal{J}(f)} & \mathcal{J}(t') \\ \uparrow s & & \uparrow s' \\ \text{dom } s & \xrightarrow{g} & \text{dom } s' \end{array}$$

commutes.

2.2. The category \mathbf{R} of relations and relation-preserving morphisms. Objects are all pairs (t, R) where t is a \mathbf{T} -object and R is a relation on $\text{hom}_{\mathbf{T}}(t, \Delta)$. Arrows $(t, R) \rightarrow (t', R')$ are the \mathbf{T} -arrows $f: t \rightarrow t'$ such that the function

$$- \circ f: \text{hom}_{\mathbf{T}}(t', \Delta) \rightarrow \text{hom}_{\mathbf{T}}(t, \Delta) \quad (4)$$

satisfies the property

$$(p', q') \in R' \implies (p' \circ f, q' \circ f) \in R.$$

We say in this case that f *preserves R' (with respect to R)*.

Remark 2.1. Observe that if (4) satisfies the property above, then it must factor through the equivalence relations $\overline{R'}$ and \overline{R} generated by R' and R , respectively. In other words, if the \mathbf{T} -arrow $f: t \rightarrow t'$ preserves R' with respect to R , then it also preserves $\overline{R'}$ with respect to \overline{R} . Hence, if f defines an \mathbf{R} -arrow $(t, R) \rightarrow (t', R')$, then it also defines an \mathbf{R} -arrow $(t, \overline{R}) \rightarrow (t', \overline{R'})$.

We emphasise that \mathbf{D} will depend on \mathcal{J} (and hence on \mathbf{T} and \mathbf{S}) but not on Δ , and \mathbf{R} will depend on Δ (and hence on \mathbf{T}) but not on \mathcal{J} (nor on \mathbf{S}). Hence a more informative notation would be $\mathbf{D}_{\mathcal{J}}$ and \mathbf{R}_{Δ} , which however we do not adopt for the sake of legibility.

Terminology. Throughout, when we say that $f: (t, s) \rightarrow (t', s')$ is a \mathbf{D} -arrow, we entail that $f: t \rightarrow t'$ is the unique \mathbf{T} -arrow that defines it. Similarly, when we say that $f: (t, R) \rightarrow (t', R')$ is an \mathbf{R} -arrow, we imply that $f: t \rightarrow t'$ is the unique \mathbf{T} -arrow that defines it.

2.3. The Galois connection (\mathbb{C}, \mathbb{V}) .

Definition 2.2. For any $(t, s) \in \mathbf{D}$, we define the following equivalence relation on $\text{hom}_{\mathbf{T}}(t, \Delta)$:

$$\mathbb{C}(s) := \{(p, q) \in \text{hom}_{\mathbf{T}}^2(t, \Delta) \mid \mathcal{J}(p) \circ s = \mathcal{J}(q) \circ s\}. \quad (5)$$

In order to define \mathbb{V} it is necessary to assume that \mathbf{S} has enough limits. It is sufficient to make the following

Assumption 1. Henceforth, we always assume that \mathbf{S} has equalisers of pairs of parallel arrows, and intersections of arbitrary families of subobjects. We denote the intersection of a family $\{E_i\}_{i \in I}$ of \mathbf{S} -subobjects by $\bigwedge_{i \in I} E_i$.

Definition 2.3. For any (t, R) in \mathbf{R} , we set

$$\mathbb{V}(R) := \bigwedge_{(p, q) \in R} \text{Eq}(\mathcal{J}(p), \mathcal{J}(q)), \quad (6)$$

where, for $(p, q) \in R$, $\text{Eq}(\mathcal{J}(p), \mathcal{J}(q))$ denotes the \mathbf{S} -subobject of $\mathcal{J}(t)$ given by the equaliser in \mathbf{S} of the \mathbf{S} -arrows $\mathcal{J}(p), \mathcal{J}(q): \mathcal{J}(t) \rightrightarrows \mathcal{J}(\Delta)$.

We now show that the operators \mathbb{V} and \mathbb{C} yield contravariant Galois connections between relations and subobjects. Let us write \leq to denote the partial order on subobjects in a category. Thus, as usual, if x and y are subobjects of z , $x \leq y$ if there is an arrow $m: \text{dom } x \rightarrow \text{dom } y$ such that $x = y \circ m$.

Lemma 2.4 (Galois connection). *For any \mathbf{T} -object t , any relation R on $\text{hom}_{\mathbf{T}}(t, \Delta)$, and any \mathbf{S} -subobject $s: \text{dom } s \rightarrow \mathcal{J}(t)$, we have*

$$R \subseteq \mathbb{C}(s) \quad \text{if, and only if,} \quad s \leq \mathbb{V}(R). \quad (7)$$

Proof. We have $R \subseteq \mathbb{C}(s)$ if, and only if, for any $(p, q) \in R$ it is the case that $\mathcal{J}(p) \circ s = \mathcal{J}(q) \circ s$. On the other hand, $s \leq \mathbb{V}(R)$ if, and only if, there is an \mathbf{S} -arrow $m: \text{dom } s \rightarrow \text{dom } \mathbb{V}(R)$ with $s = \mathbb{V}(R) \circ m$. Now, if the former holds then s must factor through $\mathbb{V}(R)$ because the latter is defined in (6) as the intersection of all \mathbf{S} -subobjects of $\mathcal{J}(t)$ that equalise some pair in R . Conversely, if the latter holds then for each $(p, q) \in R$ we obtain, composing both sides of $\mathcal{J}(p) \circ \mathbb{V}(R) = \mathcal{J}(q) \circ \mathbb{V}(R)$ with m , that $\mathcal{J}(p) \circ s = \mathcal{J}(q) \circ s$. \square

2.4. The weak Nullstellensatz. Recall that a collection of arrows $A \subseteq \text{hom}_{\mathbf{X}}(x, y)$ in a category \mathbf{X} is *jointly epic* if whenever $f_1, f_2: y \rightrightarrows z$ are \mathbf{X} -arrows with $f_1 \circ g = f_2 \circ g$ for all $g \in A$, then $f_1 = f_2$.

Theorem 2.5 (Weak Nullstellensatz). *Fix an \mathbf{R} -object (t, R) . For any family $\Sigma = \{\sigma_i\}_{i \in I}$ of subobjects of $\mathcal{J}(t)$ such that for each σ_i there exists m_i with $\sigma_i = \mathbb{V}(R) \circ m_i$ (i.e. $\sigma_i \leq \mathbb{V}(R)$) and the family of \mathbf{S} -arrows $\{m_i\}_{i \in I}$ is jointly epic in \mathbf{S} , the following are equivalent.*

- (i) $R = \mathbb{C}(\mathbb{V}(R))$, i.e. R is fixed by the Galois connection (7).
- (ii) $R = \bigcap_{i \in I} \mathbb{C}(\sigma_i)$.

Proof. First observe that the Galois connection (7) in Lemma 2.4 implies the *expansiveness* of $\mathbb{C} \circ \mathbb{V}$, i.e.

$$R \subseteq \mathbb{C}(\mathbb{V}(R)). \quad (8)$$

Further, since each $\sigma_i \leq \mathbb{V}(R)$, again by general properties of Galois connections, it follow that

$$R \subseteq \bigcap_{i \in I} \mathbb{C}(\sigma_i) . \quad (9)$$

(i) \Rightarrow (ii) As by hypothesis $R = \mathbb{C}(\mathbb{V}(R))$, by (9) above, it is enough to prove

$$\bigcap_{i \in I} \mathbb{C}(\sigma_i) \subseteq \mathbb{C}(\mathbb{V}(R)) .$$

If $(p, q) \in \bigcap_{i \in I} \mathbb{C}(\sigma_i)$, then for every $\sigma_i \in \Sigma$, $\mathcal{J}(p) \circ \sigma_i = \mathcal{J}(q) \circ \sigma_i$. By hypothesis, the latter can be rewritten as $\mathcal{J}(p) \circ \mathbb{V}(R) \circ m_i = \mathcal{J}(q) \circ \mathbb{V}(R) \circ m_i$. Now, the family of factorisations $\{m_i\}_{i \in I}$ is jointly epic in \mathbf{S} , hence we obtain $\mathcal{J}(p) \circ \mathbb{V}(R) = \mathcal{J}(q) \circ \mathbb{V}(R)$, which proves $(p, q) \in \mathbb{C}(\mathbb{V}(R))$.

(ii) \Rightarrow (i) By (8) above and the hypothesis (ii), it is enough to prove

$$\mathbb{C}(\mathbb{V}(R)) \subseteq \bigcap_{i \in I} \mathbb{C}(\sigma_i) .$$

Suppose that $(p, q) \in \mathbb{C}(\mathbb{V}(R))$, i.e. $\mathcal{J}(p) \circ \mathbb{V}(R) = \mathcal{J}(q) \circ \mathbb{V}(R)$. By composing on the right with m_i we obtain, for all $\sigma_i \in \Sigma$, $\mathcal{J}(p) \circ \mathbb{V}(R) \circ m_i = \mathcal{J}(q) \circ \mathbb{V}(R) \circ m_i$. Applying the above commutativity of σ_i one obtains $\mathcal{J}(p) \circ \sigma_i = \mathcal{J}(q) \circ \sigma_i$. The latter entails that, for all $i \in I$, $(p, q) \in \mathbb{C}(\sigma_i)$, whence $(p, q) \in \bigcap_{i \in I} \mathbb{C}(\sigma_i)$. \square

Remark 2.6. Notice that one such family Σ always exists, namely $\text{dom } \mathbb{V}(R)$ and the arrow $\mathbb{V}(R)$. However, in this case the theorem becomes tautological. When the category \mathbf{S} is **Set**, the category of sets and functions, one can chose as Σ the family of maps with domain the singleton $*$ (i.e. the terminal object of **Set**). The family Σ is obviously jointly surjective and Theorem 2.5 can be restated in a more concrete form as follows.

Theorem 2.7. *Suppose $\mathbf{S} = \mathbf{Set}$. For any \mathbf{R} -object (t, R) the following are equivalent.*

- (i) $R = \mathbb{C}(\mathbb{V}(R))$,
- (ii) $R = \bigcap_{\sigma \leq \mathbb{V}(R)} \mathbb{C}(\sigma)$, where σ ranges over all \mathbf{S} -subobjects $* \rightarrow \mathcal{J}(t)$.

The operators \mathbb{C} and \mathbb{V} naturally give rise to functors \mathcal{C} and \mathcal{V} , as we spell out in the following.

2.5. The functor $\mathcal{C}: \mathbf{D} \rightarrow \mathbf{R}$. For any \mathbf{D} -object (t, s) , we set

$$\mathcal{C}(t, s) := (t, \mathbb{C}(s)) . \quad (10)$$

For a \mathbf{D} -arrow $f: (t, s) \rightarrow (t', s')$ we let $\mathcal{C}(f)$ be the \mathbf{R} -arrow $f: (t, \mathbb{C}(s)) \rightarrow (t', \mathbb{C}(s'))$. To check that this is well-defined, we need to show that the function

$$- \circ f: \text{hom}_{\mathbf{T}}(t', \Delta) \rightarrow \text{hom}_{\mathbf{T}}(t, \Delta)$$

satisfies $(p' \circ f, q' \circ f) \in \mathbb{C}(s)$ for any $(p', q') \in \mathbb{C}(s')$. Indeed, note that for some \mathbf{S} -arrow $g: \text{dom } s \rightarrow \text{dom } s'$

$$\mathcal{J}(f) \circ s = s' \circ g , \quad (11)$$

because $f: (t, s) \rightarrow (t', s')$ is a \mathbf{D} -arrow. Now, given $p', q' \in \text{hom}_{\mathbf{T}}(t', \Delta)$, assume $(p', q') \in \mathbb{C}(s')$, that is

$$\mathcal{J}(p') \circ s' = \mathcal{J}(q') \circ s' . \quad (12)$$

Composing both sides of (12) with g , and applying (11), we obtain $\mathcal{J}(p' \circ f) \circ s = \mathcal{J}(q' \circ f) \circ s$, which shows $(p' \circ f, q' \circ f) \in \mathbb{C}(s)$.

2.6. The functor $\mathcal{V}: \mathbf{R} \rightarrow \mathbf{D}$. For any \mathbf{R} -object (t, R) , we set

$$\mathcal{V}(t, R) := (t, \mathbb{V}(R)). \quad (13)$$

For an \mathbf{R} -arrow $f: (t, R) \rightarrow (t', R')$ we define $\mathcal{V}(f)$ to be the \mathbf{D} -arrow $f: (t, \mathbb{V}(R)) \rightarrow (t', \mathbb{V}(R'))$. To check that this is well-defined, we need to show that $\mathcal{J}(f) \circ \mathbb{V}(R)$ factors through $\mathbb{V}(R')$. Indeed, let $p', q' \in \text{hom}_{\mathbf{T}}(t', \Delta)$, and assume $(p', q') \in R'$. Then $(p' \circ f, q' \circ f) \in R$ because f is an \mathbf{R} -arrow, and therefore $\mathcal{J}(p') \circ (\mathcal{J}(f) \circ \mathbb{V}(R)) = \mathcal{J}(q') \circ (\mathcal{J}(f) \circ \mathbb{V}(R))$ for all $(p', q') \in R'$. By the universal property of the pull-back $\mathbb{V}(R') := \bigwedge_{(p', q') \in R'} \text{Eq}(\mathcal{J}(p'), \mathcal{J}(q'))$ it follows that $\mathcal{J}(f) \circ \mathbb{V}(R)$ factors through $\mathbb{V}(R')$.

2.7. The weak adjunction. The Galois connection (7) lifts to an adjunction.

Theorem 2.8 (Weak affine adjunction). *The functor $\mathcal{C}: \mathbf{D} \rightarrow \mathbf{R}$ is left adjoint to the functor $\mathcal{V}: \mathbf{R} \rightarrow \mathbf{D}$. In symbols, $\mathcal{C} \dashv \mathcal{V}$.*

Proof. Let us show that for any \mathbf{D} -object (t, s) and any \mathbf{R} -object (t', R') we have a natural bijective correspondence between the \mathbf{R} -arrows $\mathcal{C}(t, s) = (t, \mathbb{C}(s)) \rightarrow (t', R')$ and the \mathbf{D} -arrows $(t, s) \rightarrow (t', \mathbb{V}(R')) = \mathcal{V}(t', R')$.

Let $f: t \rightarrow t'$ be a \mathbf{T} -arrow. Then f defines an \mathbf{R} -arrow $(t, \mathbb{C}(s)) \rightarrow (t', R')$ if, and only if, for any $p', q' \in \text{hom}_{\mathbf{T}}(t', \Delta)$, if $(p', q') \in R'$ then $\mathcal{J}(p') \circ \mathcal{J}(f) \circ s = \mathcal{J}(q') \circ \mathcal{J}(f) \circ s$. On the other hand, f defines a \mathbf{D} -arrow $(t, s) \rightarrow (t', \mathbb{V}(R'))$ in \mathbf{D} if, and only if, $f \circ s$ factors through $\mathbb{V}(R')$, i.e. for any $p', q' \in \text{hom}_{\mathbf{T}}(t', \Delta)$, if $(p', q') \in R'$ then $\mathcal{J}(p') \circ \mathcal{J}(f) \circ s = \mathcal{J}(q') \circ \mathcal{J}(f) \circ s$. It is thereby clear that $\mathcal{C} \dashv \mathcal{V}$. \square

3. THE GENERAL ADJUNCTION

We now consider appropriate quotients of the categories \mathbf{D} and \mathbf{R} . We shall need a lemma about factorisations of adjoint pairs through quotient categories [36, II.8]. A *congruence relation* on a category \mathbf{X} is a family R of equivalence relations $R_{x, x'}$ on $\text{hom}_{\mathbf{X}}(x, x')$ indexed by the pairs (x, x') of \mathbf{X} -objects, such that for all \mathbf{X} -arrows $f_1, g_1: x \rightarrow x'$, $f_2, g_2: x' \rightarrow x''$, if $(f_1, g_1) \in R_{x, x'}$ and $(f_2, g_2) \in R_{x', x''}$ then $(f_2 \circ f_1, g_2 \circ g_1) \in R_{x, x''}$. The *quotient category* \mathbf{X}/R of \mathbf{X} modulo the congruence R has then as objects the \mathbf{X} -objects, and as hom-sets the quotients $\text{hom}_{\mathbf{X}/R}(x, x') := (\text{hom}_{\mathbf{X}}(x, x'))/R_{(x, x')}$ for each pair of (\mathbf{X}/R) -objects (x, x') ; composition is defined in the obvious manner. There is a canonical projection functor

$$\mathcal{F}_R: \mathbf{X} \rightarrow \mathbf{X}/R$$

that acts identically on \mathbf{X} -objects, and carries the \mathbf{X} -arrow $x \rightarrow x'$ to the \mathbf{X}/R arrow given by its $R_{x, x'}$ -equivalence class. The functor \mathcal{F}_R is universal amongst functors $\mathcal{G}: \mathbf{X} \rightarrow \mathbf{C}$ with the property that $(f, g) \in R_{x, x'}$ implies $\mathcal{G}(f) = \mathcal{G}(g)$; see [36, Proposition II.8.1].

Lemma 3.1. *Let $\mathcal{F}: \mathbf{X} \rightarrow \mathbf{Y}$ and $\mathcal{G}: \mathbf{Y} \rightarrow \mathbf{X}$ be two adjoint functors with $\mathcal{F} \dashv \mathcal{G}$. Let R and S be two congruence relations on \mathbf{X} and \mathbf{Y} , respectively. Suppose that \mathcal{F} preserves R , in the sense that $(f, g) \in R_{x, x'}$ implies $(\mathcal{F}(f), \mathcal{F}(g)) \in S_{\mathcal{F}(x), \mathcal{F}(x')}$ for all pairs $f, g: x \rightarrow x'$ of \mathbf{X} -arrows. Similarly, suppose that \mathcal{G} preserves S . Then the factorisations $\mathcal{F}^q: \mathbf{X}/R \rightarrow \mathbf{Y}/S$ and $\mathcal{G}^q: \mathbf{Y}/S \rightarrow \mathbf{X}/R$ of \mathcal{F} and \mathcal{G} , respectively,*

through the canonical projection functors $\mathcal{F}_R: \mathbf{X} \rightarrow \mathbf{X}/R$ and $\mathcal{G}_S: \mathbf{Y} \rightarrow \mathbf{Y}/S$ are adjoint with $\mathcal{F}^q \dashv \mathcal{G}^q$.

Proof. Consider an \mathbf{X} -object x and a \mathbf{Y} -object y . Because $\mathcal{F} \dashv \mathcal{G}$, there is a natural bijection $\text{hom}_{\mathbf{Y}}(\mathcal{F}(x), y) \cong \text{hom}_{\mathbf{X}}(x, \mathcal{G}(y))$. Since \mathcal{F} and \mathcal{G} preserve R and S , respectively, it is elementary to verify that there is an induced natural bijection between the quotient sets $\text{hom}_{\mathbf{Y}/S}(\mathcal{F}^q(x), y)$ and $\text{hom}_{\mathbf{X}/R}(x, \mathcal{G}^q(y))$. Indeed, the arrow $\alpha_f: x \rightarrow \mathcal{G}(y)$ corresponding to an arrow $f: \mathcal{F}(x) \rightarrow y$ under the bijection above is given by the composite of $\mathcal{G}(f)$ with the unit $\eta_x: x \rightarrow \mathcal{G}(\mathcal{F}(x))$ of the adjunction between \mathcal{F} and G , and hence if $(f, f') \in R$ then $(\alpha_f, \alpha_{f'}) \in S$ since $\alpha_f = \mathcal{G}(f) \circ \eta_x$ and $\alpha_{f'} = \mathcal{G}(f') \circ \eta_x$ (here we use the fact that S is a congruence and that \mathcal{G} preserves R); the proof of the other direction is entirely analogous (it uses the counit of the adjunction between \mathcal{F} and \mathcal{G} , the fact that R is a congruence and the fact that \mathcal{F} preserves S). \square

Definition 3.2. We define \mathbf{D}^q to be the quotient of \mathbf{D} modulo the congruence δ defined by declaring the \mathbf{D} -arrows $f, g: (t, s) \rightarrow (t', s')$ equivalent precisely when $\mathcal{J}(f) \circ s = \mathcal{J}(g) \circ s$.

It is an exercise to check that the relation above is indeed a congruence. Specularly,

Definition 3.3. We define \mathbf{R}^q to be the quotient of \mathbf{R} modulo the congruence ρ defined by declaring the \mathbf{R} -arrows $f, g: (t, R) \rightarrow (t', R')$ equivalent precisely when the factorisations of $- \circ f, - \circ g: \text{hom}_{\mathbf{T}}(t', \Delta) \rightrightarrows \text{hom}_{\mathbf{T}}(t, \Delta)$ through the quotient sets $\text{hom}_{\mathbf{T}}(t', \Delta)/\overline{R'}$ and $\text{hom}_{\mathbf{T}}(t, \Delta)/\overline{R}$ are equal, where $\overline{R'}$ and \overline{R} denote the equivalence relations generated by R' and R , respectively. (Recall Remark 2.1.)

Once more, it is elementary to verify that the relation defined above is indeed a congruence. We therefore have canonical projection functors $\mathcal{C}_\delta: \mathbf{D} \rightarrow \mathbf{D}^q$ and $\mathcal{V}_\rho: \mathbf{R} \rightarrow \mathbf{R}^q$.

Remark 3.4. The arrows in the categories \mathbf{D}^q and \mathbf{R}^q admit the following more concrete descriptions: the arrows $(t, s) \rightarrow (t', s')$ in \mathbf{D}^q are precisely the functions $\text{dom}(s) \rightarrow \text{dom}(s')$ which are restrictions of some arrow of the form $\mathcal{J}(f)$ where $f: t \rightarrow t'$, while the arrows $(t, R) \rightarrow (t', R')$ in \mathbf{R}^q are precisely the functions $\text{hom}_{\mathbf{T}}(t', \Delta)/\overline{R'} \rightarrow \text{hom}_{\mathbf{T}}(t, \Delta)/\overline{R}$ which are induced by an arrow $t \rightarrow t'$ in the sense specified above. In fact, in Remark 4.4 we shall define functors that, under the hypotheses of Theorem 4.8, realise $\mathbf{R}^{q\text{op}}$ and \mathbf{D}^q (in case $\mathbf{S} = \mathbf{Set}$) as concrete categories structured over \mathbf{Set} .

Definition 3.5. We say that the \mathbf{T} -object Δ is an \mathcal{J} -coseparator if for any \mathbf{T} -object t the family of arrows $\mathcal{J}(\varphi): \mathcal{J}(t) \rightarrow \mathcal{J}(\Delta)$, as φ ranges over all \mathbf{T} -arrows $\varphi: t \rightarrow \Delta$, is jointly monic in \mathbf{S} . In other words, given any two \mathbf{S} -arrows $h_1, h_2: \mathcal{J}(t) \rightarrow \mathcal{J}(\Delta)$, if $\mathcal{J}(\varphi) \circ h_1 = \mathcal{J}(\varphi) \circ h_2$ for all φ , then $h_1 = h_2$.

Lemma 3.6. (i) The functor $\mathcal{C}: \mathbf{D} \rightarrow \mathbf{R}$ preserves δ . (ii) If the \mathbf{T} -object Δ is an \mathcal{J} -coseparator, the functor $\mathcal{V}: \mathbf{R} \rightarrow \mathbf{D}$ preserves ρ .

Proof. (i) Let $x = (t, s), x' = (t', s')$ be \mathbf{D} -objects, and let $f, g \in \text{hom}(x, x')$ be such that $(f, g) \in \delta_{x, x'}$. Explicitly,

$$\mathcal{J}(f) \circ s = \mathcal{J}(g) \circ s. \quad (14)$$

We need to show that $(\mathcal{C}(f), \mathcal{C}(g)) \in \rho_{\mathcal{C}(x), \mathcal{C}(x')}$. Since $\mathcal{C}(x) = (t, \mathbb{C}(s))$, $\mathcal{C}(x') = (t', \mathbb{C}(s'))$, $\mathcal{C}(f) = f$, and $\mathcal{C}(g) = g$, we equivalently need to show that the factorisations of $- \circ f, - \circ g: \text{hom}_{\mathbb{T}}(t', \Delta) \rightrightarrows \text{hom}_{\mathbb{T}}(t, \Delta)$ through the quotient sets $\text{hom}_{\mathbb{T}}(t', \Delta)/\overline{\mathbb{C}(s')}$ and $\text{hom}_{\mathbb{T}}(t, \Delta)/\overline{\mathbb{C}(s)}$ are equal. This is, for all $\varphi \in \text{hom}(t', \Delta)$ the equality $(\varphi \circ f)/\overline{\mathbb{C}(s)} = (\varphi \circ g)/\overline{\mathbb{C}(s)}$ holds, or equivalently, $((\varphi \circ f), (\varphi \circ g)) \in \mathbb{C}(s)$. The latter means by definition, cf. (5), that $\mathcal{J}(\varphi \circ f) \circ s = \mathcal{J}(\varphi \circ g) \circ s$, which can be obtained from (14) above by composing both sides with $\mathcal{J}(\varphi)$.

(ii) Let $(t, R), (t', R')$ be \mathbb{R} -objects, and suppose that $(f, g) \in \rho_{(t, R), (t', R')}$. This holds if, and only if, for all $\varphi: t' \rightarrow \Delta$, $(\varphi \circ f, \varphi \circ g) \in \overline{R}$, which in turn entails $\mathcal{J}(\varphi \circ f) \circ \mathbb{V}(R) = \mathcal{J}(\varphi \circ g) \circ \mathbb{V}(R)$, by the definition (6) of $\mathbb{V}(R)$ as an intersection of equalisers. Since Δ is an \mathcal{J} -coseparator we conclude that $\mathcal{J}(f) \circ \mathbb{V}(R) = \mathcal{J}(g) \circ \mathbb{V}(R)$. Recalling that $\mathcal{V}(f) = f$ and $\mathcal{V}(g) = g$, the last equality holds precisely when $(\mathcal{V}(f), \mathcal{V}(g)) \in \delta_{\mathcal{V}(y), \mathcal{V}(y')}$. \square

Assumption 2. In light of Lemma 3.6.ii, we henceforth assume that the \mathbb{T} -object Δ is an \mathcal{J} -coseparator.

Definition 3.7. We let $\mathcal{C}^q: \mathbb{D}^q \rightarrow \mathbb{R}^q$ and $\mathcal{V}^q: \mathbb{R}^q \rightarrow \mathbb{D}^q$ be the functors given by Lemma 3.6 as the canonical factorisations of the functors $\mathcal{V}_\rho \circ \mathcal{C}$ and $\mathcal{C}_\delta \circ \mathcal{V}$ through the projection functors \mathcal{C}_δ and \mathcal{V}_ρ , respectively.

$$\begin{array}{ccc}
 & \mathcal{V} & \\
 \mathbb{D} & \begin{array}{c} \xleftarrow{\quad} \\ \top \\ \xrightarrow{\quad} \end{array} & \mathbb{R} \\
 & \mathcal{C} & \\
 \mathcal{C}_\delta \downarrow & & \downarrow \mathcal{V}_\rho \\
 \mathbb{D}^q & \begin{array}{c} \xleftarrow{\quad} \\ \top \\ \xrightarrow{\quad} \end{array} & \mathbb{R}^q \\
 & \mathcal{C}^q &
 \end{array}$$

Theorem 3.8 (General affine adjunction). *Under our standing Assumption 2, the functors $\mathcal{C}^q: \mathbb{D}^q \rightarrow \mathbb{R}^q$ and $\mathcal{V}^q: \mathbb{R}^q \rightarrow \mathbb{D}^q$ satisfy $\mathcal{C}^q \dashv \mathcal{V}^q$.*

Proof. Combine Lemma 3.6 and Lemma 3.1. \square

Remark 3.9. The algebraic Nullstellensatz (Theorem 2.5) applies *verbatim* to the quotient categories \mathbb{D}^q and \mathbb{R}^q , too. Indeed, the theorem does not mention morphisms in \mathbb{D} or \mathbb{R} at all.

4. FURTHER GENERAL THEORY

4.1. Comprehensiveness of the framework. We prove in this section that any duality meeting Assumption 1 is amenable to our framework.

Theorem 4.1. *Suppose a duality between two categories \mathbb{X} and \mathbb{Y} is given by functors $\mathcal{F}: \mathbb{X}^{\text{op}} \rightarrow \mathbb{Y}$ and $\mathcal{G}: \mathbb{Y} \rightarrow \mathbb{X}^{\text{op}}$, further suppose that \mathbb{Y} satisfies the requirements in Assumption 1. Then there exist categories \mathbb{T}, \mathbb{S} a functor $\mathcal{J}: \mathbb{T} \rightarrow \mathbb{S}$ and an object Δ in \mathbb{T} such that:*

- (1) \mathbb{X}^{op} is equivalent to a full subcategory \mathbb{R}_{eq} of \mathbb{R} ,
- (2) \mathbb{Y} is equivalent to a full subcategory \mathbb{D}_{eq} of \mathbb{D} ,

(3) the suitable compositions of the above equivalencies with \mathcal{V} and \mathcal{C} yield \mathcal{F} and \mathcal{G} .

Proof. Let us set

- $\mathsf{T} := \mathsf{X}^{\text{op}}$,
- $\mathsf{S} := \mathsf{Y}$,
- $\mathcal{J} : \mathsf{T} \rightarrow \mathsf{S}$ equals to $\mathcal{F} : \mathsf{X}^{\text{op}} \rightarrow \mathsf{Y}$, and
- Δ a fixed but arbitrary element of X .

The categories R and D are defined as in sections 2.1 and 2.2. Define the full subcategories R_{eq} and D_{eq} as follows. The former is given by the pairs (t, R) in R such that R is the identity relation on $\text{hom}(t, \Delta)$, which we call id_t . The latter is given by the pairs (t, s) in D such that s is the identity (subobject) of $\mathcal{J}(t)$, which we call 1_t . Notice that if $f : t \rightarrow t'$ is *any* arrow in T , then f is also an arrow in R_{eq} . Indeed, $(p, q) \in \text{id}_{t'}$ iff $p = q$ and this implies $p \circ f = q \circ f$, i.e. $(p \circ f, q \circ f) \in \text{id}_t$. Further, f is also an arrow in D_{eq} for similarly trivial reasons.

It is useful to calculate how \mathcal{V} and \mathcal{C} operate on R_{eq} and D_{eq} . Given an object (t, id_t) in R_{eq} , we have $\mathcal{V}(t, \text{id}_t) = (t, \mathbb{V}(\text{id}_t))$, where

$$\mathbb{V}(\text{id}_t) = \bigwedge_{(p, q) \in \text{id}_t} Eq(\mathcal{J}(p), \mathcal{J}(q)) = \mathcal{J}(t) = 1_t .$$

Given an object $(t, 1_t)$ in D_{eq} , we have $\mathcal{C}(t, 1_t) = (t, \mathbb{C}(1_t))$, where

$$\mathbb{C}(1_t) = \{(p, q) \in \text{hom}_{\mathsf{T}}(t, \Delta) \mid 1_t \circ \mathcal{J}(p) = 1_t \circ \mathcal{J}(q)\} \quad (15)$$

$$= \{(p, q) \in \text{hom}_{\mathsf{T}}(t, \Delta) \mid \mathcal{J}(p) = \mathcal{J}(q)\} \quad (16)$$

$$= \{(p, q) \in \text{hom}_{\mathsf{T}}(t, \Delta) \mid p = q\} \quad (17)$$

$$= \text{id}_t . \quad (18)$$

where (16) holds because 1_t is an iso and (17) because $\mathcal{J} = \mathcal{F}$ is faithful. In fact, \mathcal{V} and \mathcal{C} induce an equivalence between R_{eq} and D_{eq} .

We now define four functors as follows:

- (1) $\mathcal{U}_{\mathsf{R}_{eq}} : \mathsf{R}_{eq} \rightarrow \mathsf{X}^{\text{op}}$, is defined on objects as $\mathcal{U}_{\mathsf{R}_{eq}}(t, \text{id}_t) := t$ and as the identity on arrows.
- (2) $\mathcal{U}_{\mathsf{D}_{eq}} : \mathsf{D}_{eq} \rightarrow \mathsf{Y}$, is defined on objects as $\mathcal{U}_{\mathsf{D}_{eq}}(t, 1_t) := 1_t$ and on arrows as \mathcal{F} .
- (3) $\mathcal{L} : \mathsf{X}^{\text{op}} \rightarrow \mathsf{R}_{eq}$, is defined on objects as $\mathcal{L}(x) := (x, \text{id}_x)$ and as the identity on arrows
- (4) $\mathcal{R} : \mathsf{Y} \rightarrow \mathsf{D}_{eq}$, is defined on objects as $\mathcal{R}(y) := (\mathcal{G}(y), 1_{\mathcal{G}(y)})$ and on arrows as \mathcal{G} .

So we have the following diagram:

$$\begin{array}{ccc} \mathsf{X}^{\text{op}} & \xrightleftharpoons[\mathcal{G}]{\mathcal{F}} & \mathsf{Y} \\ \mathcal{U}_{\mathsf{R}_{eq}} \updownarrow \mathcal{L} & & \mathcal{R} \updownarrow \mathcal{U}_{\mathsf{D}_{eq}} \\ \mathsf{R}_{eq} & \xrightleftharpoons[\mathcal{C}]{\mathcal{V}} & \mathsf{D}_{eq} \end{array}$$

Notice that the pairs $\mathcal{L}, \mathcal{U}_{\mathsf{R}_{eq}}$ and $\mathcal{R}, \mathcal{U}_{\mathsf{D}_{eq}}$ are equivalences, indeed:

- for any (t, id_t) in \mathbf{R}_{eq} , $\mathcal{L} \circ \mathcal{U}_{\mathbf{R}_{eq}}(t, \text{id}_t) = (t, \text{id}_t)$ and for any x in \mathbf{X} $\mathcal{U}_{\mathbf{R}_{eq}} \circ \mathcal{L}(x) = x$;
- for any $(t, 1_t)$ in \mathbf{D}_{eq} , $\mathcal{R} \circ \mathcal{U}_{\mathbf{D}_{eq}}(t, 1_t) = \mathcal{R}(1_t) = \mathcal{R}(\mathcal{I}(t)) = \mathcal{R}(\mathcal{F}(t)) = (\mathcal{G}(\mathcal{F}(t)), 1_{\mathcal{G}(\mathcal{F}(t))})$ and for any y in \mathbf{Y} $\mathcal{U}_{\mathbf{D}_{eq}} \circ \mathcal{R}(y) = \mathcal{U}_{\mathbf{D}_{eq}}(\mathcal{G}(y), 1_{\mathcal{G}(y)}) = 1_{\mathcal{G}(y)} = \mathcal{F}(\mathcal{G}(y))$.

Finally we calculate the compositions:

- For any $x \in \mathbf{X}$, $\mathcal{U}_{\mathbf{D}_{eq}} \circ \mathcal{V} \circ \mathcal{L}(x) = \mathcal{U}_{\mathbf{D}_{eq}} \circ \mathcal{V}(x, \text{id}_x) = \mathcal{U}_{\mathbf{D}_{eq}}(x, 1_x) = 1_x = \mathcal{F}(x)$
- For any $y \in \mathbf{Y}$, $\mathcal{U}_{\mathbf{R}_{eq}} \circ \mathcal{C} \circ \mathcal{R}(y) = \mathcal{U}_{\mathbf{R}_{eq}} \circ \mathcal{C}(\mathcal{G}(y), 1_{\mathcal{G}(y)}) = \mathcal{U}_{\mathbf{R}_{eq}}(\mathcal{G}(y), \text{id}_{\mathcal{G}(y)}) = \mathcal{G}(y)$.

□

4.2. The case of a representable $\mathcal{I} : \mathbf{T} \rightarrow \mathbf{S}$. In this section we show that, under the hypothesis that the functor $\mathcal{I} : \mathbf{T} \rightarrow \mathbf{S}$ is representable, a suitable restriction (spelled out in Remark 4.6) of the adjunction of Theorem 3.8 is induced by a dualising object, in the sense of the definition below.

Recall that a functor \mathcal{F} from a category \mathbf{C} into \mathbf{Set} is called *representable* if there exists an object $c \in \mathbf{C}$ such that \mathcal{F} is naturally isomorphic to the functor $\text{hom}(c, -)$. A representation of \mathcal{F} is a pair (c, ψ) where $\psi : \text{hom}(c, -) \rightarrow \mathcal{F}$ is a natural isomorphism.

Definition 4.2. Let \mathbf{A} and \mathbf{B} be two categories equipped with functors $\mathcal{U}_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{Set}$ and $\mathcal{U}_{\mathbf{B}} : \mathbf{B}^{\text{op}} \rightarrow \mathbf{Set}$. We say that an adjunction $\mathcal{F} : \mathbf{A} \rightarrow \mathbf{B}^{\text{op}}$ and $\mathcal{G} : \mathbf{B}^{\text{op}} \rightarrow \mathbf{A}$, is induced by a dualising object if there exist objects $a \in \mathbf{A}$ and $b \in \mathbf{B}$ such that

- (1) $\mathcal{U}_{\mathbf{A}}$ is representable by a ,
- (2) $\mathcal{U}_{\mathbf{B}}$ is representable by b ,
- (3) the composite functor $\mathcal{U}_{\mathbf{B}} \circ \mathcal{F}$ is represented by the object $\mathcal{G}(b)$,
- (4) the composite functor $\mathcal{U}_{\mathbf{A}} \circ \mathcal{G}$ is represented by the object $\mathcal{F}(a)$, and
- (5) $\mathcal{U}_{\mathbf{B}}(\mathcal{F}(a)) = \mathcal{U}_{\mathbf{A}}(\mathcal{G}(b))$.

Remark 4.3. If the functors \mathcal{F} and \mathcal{G} define a categorical equivalence between \mathbf{A} and \mathbf{B} , then such equivalence is induced by a dualising object if and only if conditions (1), (2), and (5) hold, for in this case conditions (3) and (4) follow from the other ones.

Remark 4.4. Notice that we can always define a forgetful functor $\mathcal{U}_{\mathbf{R}^q} : \mathbf{R}^{q\text{op}} \rightarrow \mathbf{Set}$ as follows. For any object (t, R) of \mathbf{R}^q , we set

$$\mathcal{U}_{\mathbf{R}^q}((t, R)) := \text{hom}_{\mathbf{T}}(t, \Delta) / \overline{R}$$

and for any arrow $f : (t, R) \rightarrow (t', R')$ in \mathbf{R}^q we set

$$\mathcal{U}_{\mathbf{R}^q}(f) := f' ,$$

where $f' : \text{hom}_{\mathbf{T}}(t', \Delta) / \overline{R'} \rightarrow \text{hom}_{\mathbf{T}}(t, \Delta) / \overline{R}$ is the factorisation of

$$- \circ f : \text{hom}_{\mathbf{T}}(t', \Delta) \rightarrow \text{hom}_{\mathbf{T}}(t, \Delta)$$

across the quotients $\text{hom}_{\mathbf{T}}(t', \Delta) / \overline{R'}$ and $\text{hom}_{\mathbf{T}}(t, \Delta) / \overline{R}$ (recall Remark 2.1). The fact that this functor is faithful comes immediately from the fact if two arrows yield the same factorisation, then they are equal in \mathbf{R}^q .

If, in addition, $\mathbf{S} = \mathbf{Set}$, we can define another forgetful functor $\mathcal{U}_{\mathbf{D}^q} : \mathbf{D}^q \rightarrow \mathbf{Set}$ as follows. For any $(t, s) \in \mathbf{D}^q$, we set

$$\mathcal{U}_{\mathbf{D}^q}((t, s)) := \text{dom}(s) ,$$

and for any arrow $f : (t, s) \rightarrow (t', s')$ in \mathbf{D}^q we set

$$\mathcal{U}_{\mathbf{D}^q}(f) := g ,$$

where $g : \text{dom}(s) \rightarrow \text{dom}(s')$ is the factorisation of $\mathcal{J}(f) \circ s$ through s' , as required in section 2.1. The functor $\mathcal{U}_{\mathbf{D}^q}$ is faithful, for if f' and f'' yield the same factorisation, then they are the same arrow in \mathbf{D}^q .

For the sequel, it is important to introduce the following notion.

Definition 4.5. Let (t, R) be an object of the category \mathbf{R}^q . We say that the relation R is \triangle -stable if for any $(h, k) \in R$ and any arrow $f : \triangle \rightarrow \triangle$ in \mathbf{T} , $(f \circ h, f \circ k) \in R$.

Remark 4.6. Notice that the functor $\mathcal{C}^q : \mathbf{D}^q \rightarrow \mathbf{R}^q$ takes values in the full subcategory \mathbf{R}_s^q of \mathbf{R}^q on the objects (t, R) such that R is \triangle -stable. Therefore, if we denote by $\mathcal{C}_s^q : \mathbf{D}^q \rightarrow \mathbf{R}_s^q$ this restriction of the functor \mathcal{C}^q , and by $\mathcal{V}_s^q : \mathbf{R}_s^q \rightarrow \mathbf{D}^q$ the restriction of the functor $\mathcal{V}^q : \mathbf{R}^q \rightarrow \mathbf{D}^q$ to the subcategory \mathbf{R}_s^q , we have that the adjunction of Theorem 3.8 restricts to an adjunction $\mathcal{C}_s^q \dashv \mathcal{V}_s^q$. We shall denote by \mathbf{D}_i^q the full subcategory of \mathbf{D}^q whose objects are (up to isomorphism) of the form $\mathcal{V}_s^q(t, R)$ for some object (t, R) of \mathbf{R}_s^q , and by $\mathcal{C}_i^q : \mathbf{D}_i^q \rightarrow \mathbf{R}_s^q$ and $\mathcal{V}_i^q : \mathbf{R}_s^q \rightarrow \mathbf{D}_i^q$ the restricted functors; then we clearly have an adjunction $\mathcal{C}_i^q \dashv \mathcal{V}_i^q$.

In the following let us denote by id_\triangle the identical relation on the set $\text{hom}_{\mathbf{T}}(\triangle, \triangle)$.

Lemma 4.7. *The functor $\mathcal{U}_{\mathbf{R}^q} : \mathbf{R}^{q\text{op}} \rightarrow \mathbf{Set}$ is represented by the object $(\triangle, \text{id}_\triangle)$ of \mathbf{R}^q .*

Proof. We need to provide a set-theoretical bijection $\text{hom}_{\mathbf{R}^q}((t, R), (\triangle, \text{id}_\triangle)) \cong \text{hom}_{\mathbf{T}}(t, \triangle) / \overline{R}$ naturally in $(t, R) \in \mathbf{R}^q$. By Definition 3.3, the arrows $(t, R) \rightarrow (\triangle, \text{id}_\triangle)$ in \mathbf{R}^q are the arrows $f : t \rightarrow \triangle$ in \mathbf{T} (as all of them preserve id_\triangle), modulo the equivalence relation ρ . Recall from Definition 3.3 that $f \rho f'$ if, and only if, the factorisations of $- \circ f, - \circ f'$ through $\text{hom}_{\mathbf{T}}(\triangle, \triangle) / \text{id}_\triangle$ and $\text{hom}_{\mathbf{T}}(t, \triangle) / \overline{R}$ (which by an abuse of notation we still indicate by $- \circ f$ and $- \circ f'$) are equal. This latter condition is equivalent to saying that $(f, f') \in \overline{R}$. Indeed, if $- \circ f = - \circ f'$ then $(- \circ f)(1_\triangle) = (- \circ f')(1_\triangle)$ in $\text{hom}_{\mathbf{T}}(t, \triangle) / \overline{R}$, i.e. $(f, f') \in \overline{R}$. For the other implication, notice that, by assumption, R is \triangle -stable. This amounts to saying that, for any arrow $g \in \text{hom}_{\mathbf{T}}(\triangle, \triangle)$, $(g \circ f, g \circ f') \in \overline{R}$, hence $(- \circ f)(g)$ is equal to $(- \circ f')(g)$ in $\text{hom}_{\mathbf{T}}(t, \triangle) / \overline{R}$. This proves that sending a \mathbf{R}^q -arrow from (t, R) to $(\triangle, \text{id}_\triangle)$ into its \overline{R} -equivalence class in $\text{hom}_{\mathbf{T}}(t, \triangle)$, gives a bijection. \square

Theorem 4.8. *Under our standing Assumption 2 that \triangle is an \mathcal{J} -coseparator, if the category \mathbf{S} coincides with \mathbf{Set} and the functor $\mathcal{J} : \mathbf{T} \rightarrow \mathbf{S}$ is represented by an object ∇ then the adjunction $\mathcal{C}_i^q \dashv \mathcal{V}_i^q$ of Remark 4.6 is induced by a dualising object.*

Proof. Notice that the functor $\mathcal{U}_{\mathbf{R}^q} : \mathbf{R}^{q\text{op}} \rightarrow \mathbf{Set}$ is represented by the object $(\triangle, \text{id}_\triangle)$ of \mathbf{R}^q by Lemma 4.7. Let us denote by 1_∇ the identical subobject of $\mathcal{J}(\nabla)$.

Claim (1) The functor $\mathcal{U}_{\mathbf{D}_i^q} : \mathbf{D}_i^q \rightarrow \mathbf{Set}$ is represented by the object $(\nabla, 1_\nabla)$ of \mathbf{D}^q . A natural isomorphism from $\text{hom}((\nabla, 1_\nabla), -)$ to $\mathcal{U}_{\mathbf{D}^q}$ amounts to a set-theoretical

bijection between $\text{hom}_{\mathcal{D}_i^q}((\nabla, 1_\nabla), (t, s))$ and $\text{dom}(s)$, holding naturally in $(t, s) \in \mathcal{D}_i^q$. Notice that, by definition of the category \mathcal{D}_i^q , any (t, s) in \mathcal{D}_i^q has the form $(t, \mathbb{V}(R))$, for some object (t, R) of the category \mathcal{R}_s^q . The arrows $(\nabla, 1_\nabla) \rightarrow (t, s)$ in \mathcal{D}^q are, by Definition 3.2, the arrows $f: \nabla \rightarrow t$ such that $1_\nabla \circ \mathcal{J}(f) = \mathcal{J}(f)$ factors through $s = \mathbb{V}(R)$, modulo the equivalence relation \simeq given by:

$$f \simeq f' \text{ if and only if } \mathcal{J}(f) = \mathcal{J}(f') .$$

Now, since the functor \mathcal{J} is represented by the object ∇ we have a natural isomorphism

$$\xi: \text{hom}_{\mathcal{T}}(\nabla, -) \cong \mathcal{J}, \quad (19)$$

from which it follows at once that for any $k \in \text{hom}_{\mathcal{T}}(\nabla, t)$ the following diagram commutes:

$$\begin{array}{ccc} \text{hom}_{\mathcal{T}}(\nabla, \nabla) & \xrightarrow{k \circ -} & \text{hom}_{\mathcal{T}}(\nabla, t) \\ \xi_\nabla \downarrow \cong & & \cong \downarrow \xi_t \\ \mathcal{J}(\nabla) & \xrightarrow{\mathcal{J}(k)} & \mathcal{J}(t) \end{array}$$

FIGURE 1. The naturality diagram for the isomorphism $\mathcal{J} \cong \text{hom}_{\mathcal{T}}(\nabla, -)$ with respect to k .

It follows that for any $k \in \text{hom}_{\mathcal{T}}(\nabla, t)$, $\xi_t(k) = \mathcal{J}(k)(\xi_\nabla(1_\nabla))$. Therefore for any $f, f' \in \text{hom}_{\mathcal{T}}(\nabla, t)$,

$$\mathcal{J}(f) = \mathcal{J}(f') \text{ if and only if } \xi_t(f) = \xi_t(f').$$

Indeed, $\xi_t(f) = \xi_t(f')$ implies $f = f'$ (since ξ_t is an isomorphism) and hence $\mathcal{J}(f) = \mathcal{J}(f')$, while by the naturality in t of ξ_t $\mathcal{J}(f) = \mathcal{J}(f')$ implies $\xi_t(f) = \xi_t(f')$, since $\xi_t(f) = \mathcal{J}(f)(\xi_\nabla(1_\nabla))$ and $\xi_t(f') = \mathcal{J}(f')(\xi_\nabla(1_\nabla))$. Therefore for any arrow $f: \nabla \rightarrow t$ in \mathcal{T} , we have that $\mathcal{J}(f)$ factors through s if and only if $\xi_t(f)$ lies in $\text{dom}(s)$. This can be proved as follows. We have that $\mathcal{J}(f)$ factors through s if and only if for any $(h, k) \in R$, $\mathcal{J}(h) \circ \mathcal{J}(f) = \mathcal{J}(k) \circ \mathcal{J}(f)$ (i.e. $\mathcal{J}(h \circ f) = \mathcal{J}(k \circ f)$); but, by the remarks above, this holds if and only if $\xi_\Delta(k \circ f) = \xi_\Delta(h \circ f)$ which, by the naturality of ξ with respect to the arrows k and h , is equivalent to the requirement $\mathcal{J}(k)(\xi_t(f)) = \mathcal{J}(h)(\xi_t(f))$, which holds if $\xi_t(f)$ lies in the image of $s = \mathbb{V}(R)$ (by definition of $\mathbb{V}(R)$). Conversely, if $\mathcal{J}(f)$ factors through s then *a fortiori* $\xi_t(f) = \mathcal{J}(f)(\xi_\nabla(\text{id}_\nabla)) \in \text{dom}(s)$.

The arrows $(\nabla, 1_\nabla) \rightarrow (t, \mathbb{V}(R))$ in \mathcal{D}^q can thus be naturally identified with the elements of $\mathcal{J}(t)$ which are in the image of the subobject s , i.e. with the elements of $\text{dom}(s)$, as required.

Claim (2) The composite functor $\mathcal{U}_{R^q} \circ \mathcal{C}_s^{q\text{op}}: \mathcal{D}_s^{q\text{op}} \rightarrow \mathbf{Set}$ is represented by the object $\mathcal{V}_s^q((\Delta, \text{id}_\Delta)) = (\Delta, 1_\Delta)$.

To verify this, we have to exhibit a bijection $\text{hom}_{\mathcal{D}^q}((t, s), (\Delta, 1_\Delta)) \cong \text{hom}_{\mathcal{T}}(t, \Delta) / \mathbb{C}(s)$ natural in $(t, s) \in \mathcal{D}^q$. Applying Definition 3.2, we obtain that the arrows $(t, s) \rightarrow (\Delta, 1_\Delta)$ in \mathcal{D}_q are the arrows $f: t \rightarrow \Delta$ modulo the equivalence relation \simeq

defined by $f \simeq f'$ if and only if $\mathcal{J}(f) \circ s = \mathcal{J}(f') \circ s$. By Definition 2.2, this latter condition is satisfied precisely $(f, f') \in \mathbb{C}(s)$, as required.

Claim (3) The composite functor $\mathcal{U}_{D^q} \circ \mathcal{V}_s^q : \mathbb{R}^q \rightarrow \mathbf{Set}$ is represented by the object $\mathcal{C}_s^q((\nabla, 1_\nabla)) = (\nabla, \text{id}_\nabla)$.

We have to check that there exists a bijection

$$\text{hom}_{\mathbb{R}^q}((\nabla, \text{id}_\nabla), (t, R)) \cong \text{dom}(\mathbb{V}(R)),$$

natural in $(t, R) \in \mathbb{R}^q$. By Definition 3.3, the arrows $(\nabla, \text{id}_\nabla) \rightarrow (t, R)$ in \mathbb{R}^q are the arrows $f : \nabla \rightarrow t$ such that for any $(h, k) \in R$, $f \circ h = f \circ k$, modulo the equivalence relation \simeq given by: $f \simeq f'$ if and only if the factorisations of the arrows $- \circ f, - \circ f'$ through $\text{hom}_\top(t, \Delta)/\overline{R}$ and $\text{hom}_\top(\nabla, \Delta)$ are equal. We claim that

$$f \simeq f' \text{ if and only if } f = f'. \quad (20)$$

Recall that the functor \mathcal{J} is represented by the object ∇ . So, for any arrow $g : t \rightarrow \Delta$, (19) gives the following commutative naturality square.

$$\begin{array}{ccc} \text{hom}_\top(\nabla, t) & \xrightarrow{g \circ -} & \text{hom}_\top(\nabla, \Delta) \\ \xi_t \downarrow \cong & & \cong \downarrow \xi_\Delta \\ \mathcal{J}(t) & \xrightarrow{\mathcal{J}(g)} & \mathcal{J}(\Delta) \end{array}$$

FIGURE 2. The naturality diagram for the isomorphism $\mathcal{J} \cong \text{hom}_\top(\nabla, -)$ with respect to g .

Notice that the factorisations of $- \circ f$ and $- \circ f'$ are equal if and only if for every $g \in \text{hom}_\top(t, \Delta)$, $g \circ f = g \circ f'$, if, and only if, $(g \circ -)(f) = (g \circ -)(f')$. Hence, the commutativity of the above diagram allows to rewrite the latter condition as $\mathcal{J}(g)(\xi_t(f)) = \mathcal{J}(g)(\xi_t(f'))$. Now, since Δ is a \mathcal{J} -coseparator, $\mathcal{J}(g)(\xi_t(f)) = \mathcal{J}(g)(\xi_t(f'))$ holds for all $g \in \text{hom}_\top(t, \Delta)$ if and only if $\xi_t(f) = \xi_t(f')$. In turn, ξ_t being an isomorphism, the latter is equivalent to $f = f'$. This complete the proof of (20). In order to complete the proof the claim, it remains to observe that for any $f \in \text{hom}_\top(\nabla, t)$, $\xi_t(f) \in \text{dom}(\mathbb{V}(R))$ if and only if for any $(h, k) \in R$, $h \circ f = k \circ f$. Indeed, $\xi_t(f) \in \text{dom}(\mathbb{V}(R))$ if and only if for any $(h, k) \in R$, $\mathcal{J}(k)(\xi_t(f)) = \mathcal{J}(h)(\xi_t(f))$; but, by the naturality of ξ , $\mathcal{J}(k)(\xi_t(f)) = \xi_\Delta(k \circ f)$ and $\mathcal{J}(h)(\xi_t(f)) = \xi_\Delta(h \circ f)$, whence $\mathcal{J}(k)(\xi_t(f)) = \mathcal{J}(h)(\xi_t(f))$ if and only if $\xi_\Delta(h \circ f) = \xi_\Delta(k \circ f)$ i.e., if and only if $h \circ f = k \circ f$.

To conclude the whole proof, it remains to observe that $\mathcal{U}_{D^q}((\Delta, 1_\Delta)) = \mathcal{J}(\Delta)$, $\mathcal{U}_{\mathbb{R}^q}((\nabla, \text{id}_\nabla)) = \text{hom}_\top(\nabla, \Delta)$, and that, by the representability of \mathcal{J} , $\mathcal{J}(\Delta) \cong \text{hom}_\top(\nabla, \Delta)$. \square

Remark 4.9. The functors $\mathcal{U}_{D^q} : D^q \rightarrow \mathbf{Set}$ and $\mathcal{U}_{\mathbb{R}^q} : \mathbb{R}^{q\text{op}} \rightarrow \mathbf{Set}$ defined above are faithful and representable, that is they realise D^q and $\mathbb{R}^{q\text{op}}$ as concrete categories structured over \mathbf{Set} (cf. Remark 3.4).

4.3. The setting of syntactic categories. The notion of *syntactic category* of a first-order theory is particularly useful in connection with the abstract categorical framework for generating affine adjunctions developed in Part 2. In fact, it allows us to apply this framework in contexts which go well beyond the standard setting of universal algebra investigated in Part 3. In the following paragraphs we recall just the basic notions useful for our analysis; for more details we refer the reader to an introduction to categorical logic and topos theory (see e.g., [13]).

Definition 4.10. Let \mathbb{T} be a theory over a signature \mathcal{L} in a given fragment of first-order logic. The *syntactic category* $\mathcal{C}_{\mathbb{T}}$ of \mathbb{T} has as objects the formulas-in-context $\{\vec{x}.\phi\}$ over the signature (considered up to ‘renaming’ of variables), where the context \vec{x} contains all the free variables appearing in the formula ϕ . The arrows $\{\vec{x}.\phi\} \rightarrow \{\vec{y}.\psi\}$ ³ are the \mathbb{T} -provable equivalence classes $[\theta]$ of formulas $\theta(\vec{x}, \vec{y})$ which are \mathbb{T} -provably functional from $\phi(\vec{x})$ to $\psi(\vec{y})$, in the sense that the sequents

$$\begin{aligned} &(\phi \vdash_{\vec{x}} (\exists \vec{y}) \theta(\vec{x}, \vec{y})), \\ &(\theta \vdash_{\vec{x}, \vec{y}} \phi \wedge \psi) \end{aligned}$$

and

$$(\theta(\vec{x}, \vec{y}) \wedge \theta(\vec{x}, \vec{z}/\vec{y}) \vdash_{\vec{x}, \vec{y}, \vec{z}} \vec{y} = \vec{z})$$

are provable in \mathbb{T}^4 .

The notion of \mathbb{T} -provably functional formula naturally generalises the notion of (morphism defined by) a term; indeed, for any term $t(\vec{x})$, the formula $\vec{y} = t(\vec{x})$ is provably functional from $\{\vec{x}.\top\}$ to $\{\vec{y}.\top\}$.

We shall be concerned in particular with syntactic categories of *geometric theories*, i.e. (many-sorted) first-order theories whose axioms can be presented in the form $(\forall \vec{x})(\phi(\vec{x}) \Rightarrow \psi(\vec{x}))$, where $\phi(\vec{x})$ and $\psi(\vec{x})$ are *geometric formulas* that is, first-order formulas built-up from atomic formulas by only using finitary conjunctions, possibly infinitary disjunctions and existential quantifications.

One can consider models of (many-sorted) first-order theories not only in the classical set-theoretic context, but in arbitrary Grothendieck toposes. The sorts of the language over which the theory is written are interpreted as objects of the topos, the function symbols as arrows, and the relation symbols as subobjects; the interpretation of the formulas is given by subobjects defined inductively from these data by using the categorical structure present in the topos, mimicking the classical Tarskian definition of first-order structure. Recall that the category **Set** of sets and functions between them is a Grothendieck topos.

For any geometric theory \mathbb{T} , the models of \mathbb{T} in any Grothendieck topos \mathcal{E} can be identified with functors $\mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{E}$ preserving the geometric structure on the category $\mathcal{C}_{\mathbb{T}}$. The functor $F_M : \mathcal{C}_{\mathbb{T}} \rightarrow \mathcal{E}$ corresponding to a \mathbb{T} -model M in \mathcal{E} sends $\{\vec{x}.\phi\}$ to (the domain of) its interpretation $[[\vec{x}.\phi]]_M$ in M and any arrow $[\theta] : \{\vec{x}.\phi\} \rightarrow \{\vec{y}.\psi\}$ in $\mathcal{C}_{\mathbb{T}}$ to the arrow $[[\vec{x}.\phi]]_M \rightarrow [[\vec{y}.\psi]]_M$ in \mathcal{E} , denoted by $[[\theta]]_M$ abusing notation, whose graph is the interpretation of the formula θ in M .

To apply the categorical framework of section 2, we stipulate, for a given geometric theory \mathbb{T} :

- \top a suitable full subcategory of $\mathcal{C}_{\mathbb{T}}$ containing the object $\{x.\top\}$;
- \triangle the object $\{x.\top\}$;

³We always suppose, without loss of generality, that the contexts \vec{x} and \vec{y} are disjoint.

⁴Recall that a sequent $(\phi \vdash_{\vec{x}} \psi)$ has the same meaning that the first-order sentence $(\forall \vec{x})(\phi \Rightarrow \psi)$.

- \mathcal{S} a Grothendieck topos (for instance, the category **Set** of sets);
- \mathcal{I} the functor $F_M : \mathbb{T} \rightarrow \mathcal{S}$ corresponding to an arbitrarily fixed \mathbb{T} -model M in \mathcal{S} as specified above.

Assumption 1 of Part 2 is satisfied as any Grothendieck topos \mathcal{S} has small limits. The next lemma takes care of verifying the remaining requirement that Δ is a \mathcal{I} -coseparator.

Lemma 4.11. *In the setting defined above, the object Δ is always a \mathcal{I} -coseparator.*

Proof. We have to verify that for every object $\{\vec{x}.\phi\}$ of \mathbb{T} , the family of arrows $[[\theta]]_M : [[\vec{x}.\phi]]_M \rightarrow M$, where θ varies among the \mathbb{T} -provably functional formulas from $\{\vec{x}.\phi\}$ to $\{y.\top\}$, is jointly monic in \mathcal{S} . Now, if $\vec{x} = (x_1, \dots, x_n)$, for any $i \in \{1, \dots, n\}$ the formula $y = x_i \wedge \phi(\vec{x})$ is \mathbb{T} -provably functional $\{\vec{x}.\phi\}$ to $\{y.\top\}$. But the interpretations in M of such formulas are nothing but the canonical projections $[[\vec{x}.\phi]]_M \subseteq M^n \rightarrow M$, which are obviously jointly monic. \square

Remarks 4.12. (a) If \mathbb{T} is an algebraic theory, it is natural to take \mathbb{T} equal to the subcategory of $\mathcal{C}_{\mathbb{T}}$ whose objects are the powers of $\{x.\top\}$. One can prove that the \mathbb{T} -functional formulas between formulas are all induced by terms, up to \mathbb{T} -provable equivalence (see e.g. [9, p. 120], where it is proved that any \mathbb{T} -provably functional geometric formula $\theta(\vec{x}, \vec{y})$ between Horn formulas is \mathbb{T} -provably equivalent to a formula of the form $\vec{y} = \vec{t}(\vec{x})$, where \vec{t} is a sequence of terms of the appropriate sorts in the context \vec{x}). As we shall see below, the algebraic framework of Part 3 can be precisely obtained by specialising the framework defined above to such syntactic categories.

(b) Let M be a *conservative model* for \mathbb{T} , i.e. a model of \mathbb{T} such that any assertion (in the relevant fragment of logic) over the signature of \mathbb{T} which is valid in it is provable in \mathbb{T} . Then the arrows $\{x.\top\}^k \rightarrow \{x.\top\}$ in \mathbb{T} can be identified with the \mathbb{T} -model definable homomorphisms $M^k \rightarrow M$, for each k . Indeed, for any two \mathbb{T} -provably functional formulas θ_1, θ_2 from $\{x.\top\}^k$ to $\{x.\top\}$, we have $[\theta_1] = [\theta_2]$ if and only if θ_1 and θ_2 are \mathbb{T} -provably equivalent; but this is equivalent, M being conservative, to the condition that $[[\theta_1]]_M = [[\theta_2]]_M$.

As an example, let \mathbb{T} be the algebraic theory of Boolean algebras. The algebra $\{0, 1\}$ is a conservative model for \mathbb{T} , and in fact the free Boolean algebra on k generators can be identified with the set of definable maps $\{0, 1\}^k \rightarrow \{0, 1\}$.

(c) This framework generalises that of Part 3, which relies on the existence of a free object in the variety. By working at the syntactic level, we can carry out our constructions by replacing the underlying set of each free object on k generators by the set of arrows $\{\vec{x}^k.\top\} \rightarrow \{x.\top\}$ in the syntactic category of the theory.

A particularly natural class of theories to which one can apply the setting defined above is that of theories of presheaf type.

A (geometric) theory is said to be of *presheaf type* if it is classified by a presheaf topos (for a self-contained introduction to the theory of classifying toposes we refer the reader to [13]). This class of theories is interesting for several reasons:

- (1) Every finitary algebraic theory (or, more generally, any cartesian theory) is of presheaf type;
- (2) There are many other interesting mathematical theories which are of presheaf type without being cartesian, such as the theory of total orders, the theory

of algebraic extensions of a base field, the theory of lattice-ordered abelian groups with strong unit, the theory of perfect MV-algebras, the cyclic theory classified by Connes' topos (cf. [14]) etc.

- (3) Every small category can be regarded, up to idempotent-splitting completion, as the category of finitely presentable models of a theory of presheaf type (cf. [12]).

The class of theories of presheaf type thus represents a natural generalisation of the class of algebraic theories. For a comprehensive study of this class of theories, we refer the reader to [12].

Interestingly, free objects in the category of set-based models of a theory of presheaf type \mathbb{T} do not always exist, but this category is always generated by the finitely presentable (equivalently, finitely presented) models of the theory. The full subcategory spanned by such models is dual to the full subcategory of the syntactic category of the theory \mathbb{T} on the \mathbb{T} -irreducible formulas (cf. [11]), and for each such formula $\phi(\vec{x})$ presenting a model M_ϕ , we have $M_\phi \cong \text{hom}_{\mathcal{C}_{\mathbb{T}}}(\{\vec{x}.\phi\}, \{x.\top\})$ (cf. [12]).

Definition 4.13. Let \mathbb{T} be a geometric theory and M a \mathbb{T} -model.

- (a) A *definable map* $M^k \rightarrow M$ is a map of the form $[[\theta]]_M$ where θ is a \mathbb{T} -provably functional formula from $\{x.\top\}^k$ to $\{x.\top\}$.
- (b) A *congruence* on M is an equivalence relation R on M such that for any definable map $d : M^k \rightarrow M$, $(x_i, y_i) \in R$ for all $i = 1, \dots, k$ implies that $(d(x_1, \dots, x_k), d(y_1, \dots, y_k)) \in R$.

Remark 4.14. As we recalled above, if \mathbb{T} is a finitary algebraic theory then the \mathbb{T} -provably functional formulas $\{x.\top\}^k$ to $\{x.\top\}$ are all induced by terms, up to \mathbb{T} -provable equivalence, so the above notions specialize to the classical universal algebraic ones.

Proposition 4.15. *If \mathbb{T} is a theory of presheaf type and M is a finitely presentable \mathbb{T} -model then any congruence on M is Δ -stable (regarding M as $\text{hom}_{\mathbb{T}}(\{\vec{x}.\phi\}, \Delta)$, where $\{\vec{x}.\phi\}$ is a formula presenting M).*

Proof. We have to show that if R is a congruence on $M = \text{hom}_{\mathbb{T}}(\{\vec{x}.\phi\}, \Delta)$ then for any $([\theta_1], [\theta_2]) \in R$ and any arrow $[\theta] : \Delta \rightarrow \Delta$ in \mathbf{S} , $([\theta \circ \theta_1], [\theta \circ \theta_2]) \in R$. Now, $[[\theta]]_M$ is precisely the function $[\theta] \circ - : \text{hom}_{\mathbb{T}}(\{\vec{x}.\phi\}, \Delta) \rightarrow \text{hom}_{\mathbb{T}}(\{\vec{x}.\phi\}, \Delta)$, whence our thesis follows immediately. \square

Hence, taking \mathbb{T} to be the full subcategory of the geometric syntactic category $\mathcal{C}_{\mathbb{T}}$ of a theory of presheaf type \mathbb{T} on the formulas which are either $\{y.\top\}$ or \mathbb{T} -irreducible and \mathbf{S} to be \mathbf{Set} yields in particular an adjunction between a category of congruences on finitely presentable \mathbb{T} -models and a certain category of definable sets and \mathbb{T} -definable homomorphisms between them.

We note that the equivalence between the first two items in the algebraic *Nullstellensatz* (Theorem 7.7) holds more generally for any theory of presheaf type \mathbb{T} (replacing $\mathcal{F}(\mu)$ with any finitely presentable \mathbb{T} -model), with essentially the same proof.

4.4. Recovering Diers' "system-solution" adjunction. In this section we show that Diers' system-solution adjunction ([20, Proposition 3.6]) can be recovered as an instance of Theorem 2.8.

The context in which Diers works is that of an (essentially) algebraic theory \mathbb{T} , and of a fixed model \mathcal{L} in **Set**. Diers defines a category $\mathbf{AfSubSet}(\mathcal{L})$ of affine subsets over \mathcal{L} whose objects are the triplets $(X, A(X), Y)$, where X is a set, $A(X)$ is a \mathbb{T} -subalgebra of the \mathbb{T} -algebra \mathcal{L}^X and Y is a subset of X , and whose arrows $(X, A(X), Y) \rightarrow (X', A(X'), Y')$ are the functions $f: X \rightarrow X'$ such that $F(Y) \subseteq Y'$ and $\mathcal{L}^f: \mathcal{L}^{X'} \rightarrow \mathcal{L}^X$ restricts to a function (in fact, a \mathbb{T} -model homomorphism) $A(X') \rightarrow A(X)$. On the other side, he considers a category $\mathbf{AlgSyst}(\mathcal{L})$ of algebraic systems over \mathcal{L} , whose objects are triplets $(X, A(X), E)$, where X is a set, $A(X)$ is a \mathbb{T} -subalgebra of the \mathbb{T} -algebra \mathcal{L}^X and E is a set of pairs (u, v) , where $u, v \in A(X)$. He then defines two functors defining a “system-solution” adjunction: $\mathcal{Z}: \mathbf{AlgSyst}(\mathcal{L}) \rightarrow \mathbf{AfSubSet}(\mathcal{L})$ and $\mathcal{S}: \mathbf{AfSubSet}(\mathcal{L}) \rightarrow \mathbf{AlgSyst}(\mathcal{L})$ by setting $\mathcal{Z}(X, A(X), E)$ equal to $(X, A(X), S)$, where S is the locus of solutions in X of the equations $u = v$ for $(u, v) \in E$ and $\mathcal{S}(X, A(X), Y) = (X, A(X), E)$, where E is the set of pairs (u, v) of elements of $A(X)$ such that $u(x) = v(x)$ for all $x \in Y$.

This adjunction can be recovered as a particular case of our Theorem 3.8 by setting:

- \mathbb{T} equal to the category $\mathbf{AfSet}(\mathcal{L})$ whose objects are the pairs $(X, A(X))$, where X is a set, $A(X)$ is a \mathbb{T} -subalgebra of the \mathbb{T} -algebra \mathcal{L}^X and whose arrows $(X, A(X)) \rightarrow (X', A(X'))$ are the functions $f: X \rightarrow X'$ such that $\mathcal{L}^f: \mathcal{L}^{X'} \rightarrow \mathcal{L}^X$ restricts to a function $A(X') \rightarrow A(X)$.
- \mathbf{S} equal to the category **Set** of sets
- $\mathcal{J}: \mathbb{T} \rightarrow \mathbf{S}$ equal to the forgetful functor sending any object $(X, A(X))$ in \mathbb{T} to the set X and any arrow $f: (X, A(X)) \rightarrow (Y, A(Y))$ in \mathbb{T} to the function $f: X \rightarrow Y$;
- Δ equal to the object $(\mathcal{L}, \mathcal{A}_{\mathcal{L}})$, where $\mathcal{A}_{\mathcal{L}}$ is the \mathbb{T} -subalgebra of $\mathcal{L}^{\mathcal{L}}$ generated by the set $\{1_{\mathcal{L}}\}$, where $1_{\mathcal{L}}$ is the identity on \mathcal{L} .

Indeed, for any object $(X, A(X))$ of \mathbb{T} , the set $\text{hom}_{\mathbb{T}}((X, A(X)), (\mathcal{L}, \mathcal{A}_{\mathcal{L}}))$ is canonically isomorphic to $A(X)$, since a function $f: X \rightarrow \mathcal{L}$ belongs to $A(X)$ if and only if $\mathcal{L}^f = - \circ f: \mathcal{L}^{\mathcal{L}} \rightarrow \mathcal{L}^X$ restricts to a function $\mathcal{A}_{\mathcal{L}} \rightarrow A(X)$ (note that the arrow \mathcal{L}^f is a \mathbb{T} -algebra homomorphism and hence its restriction to $\mathcal{A}_{\mathcal{L}}$ factors through $A(X) \hookrightarrow \mathcal{L}^X$ if and only if $\mathcal{L}^f(1_{\mathcal{L}}) = f \in A(X)$, as any element of $\mathcal{A}_{\mathcal{L}}$ can be obtained from $1_{\mathcal{L}}$ by applying the \mathbb{T} -operations a finite number of times).

The objects of the category \mathbf{R} can thus be identified with the pairs $((X, A(X)), R)$, where R is a relation on the set $A(X)$. The arrows $((X, A(X)), R) \rightarrow ((Y, A(Y)), R')$ are the functions $f: X \rightarrow Y$ such that \mathcal{L}^f restricts to a function $A(Y) \rightarrow A(X)$ which sends R' -pairs to R -pairs. In other words, \mathbf{R} coincides with the category $\mathbf{AlgSyst}(\mathcal{L})$. On the other hand, it is immediate to see that the category \mathbf{D} coincides with the category $\mathbf{AfSubSet}(\mathcal{L})$ of affine subsets over \mathcal{L} of [20]. It is also clear that the adjunction of Theorem 2.8 specializes precisely to the adjunction between \mathcal{Z} and \mathcal{S} of [20, Proposition 3.6].

Part 3. The specialisation to varieties of algebras

5. THE GENERAL SETTING

We now specialise the setting of part 2 to algebraic categories. In particular, we shall work with *varieties of algebras* (i.e., equationally definable classes of algebras) in the sense of Słominsky [44] and Linton [31].

In this section we fix the following notation.

- \mathbf{V} is a (possibly infinitary) variety of algebras, regarded as a category whose objects are the \mathbf{V} -algebras and whose morphisms are the \mathbf{V} -homomorphisms.
- $\mathcal{U} : \mathbf{V} \rightarrow \mathbf{Set}$ is the canonical underlying set functor.
- \mathcal{F} is the canonical free functor, i.e. the left adjoint to \mathcal{U} .
- A is an arbitrary but fixed \mathbf{V} -algebra.

We henceforth often speak of ‘algebras’ and ‘homomorphisms’ (and also ‘isomorphisms’ etc.) rather than ‘ \mathbf{V} -algebras’ and ‘ \mathbf{V} -homomorphisms’, the variety \mathbf{V} being understood.

If I is any set, the algebra $\mathcal{F}(I)$ in \mathbf{V} is, as usual, *a free algebra generated by I* . We fix canonical representatives for the isomorphism class of each free algebra in \mathbf{V} . To this end, let

$$X_\mu := \{X_\alpha\}_{\alpha < \mu} \quad (21)$$

be a specific set (of *variables*, or *free generators*) of cardinality μ , where α ranges over ordinals (of cardinality) less than μ . We often write μ as a shorthand for X_μ , and therefore $\mathcal{F}(\mu)$ as a shorthand for $\mathcal{F}(X_\mu)$. To stress that we are selecting a specific representative for the isomorphism class of a free algebra $\mathcal{F}(I)$, we refer to $\mathcal{F}(\mu)$ as *the* free algebra on μ generators.

The adjunction relation

$$\frac{\mathcal{F}(\mu) \rightarrow A}{\mu \rightarrow \mathcal{U}(A)} \quad (22)$$

shows that $\mathcal{U}(A)$ may be naturally identified (in \mathbf{Set}) with the set of homomorphisms $\mathcal{F}(1) \rightarrow A$, i.e.

$$\mathcal{U}(A) \cong \text{hom}_{\mathbf{V}}(\mathcal{F}(1), A). \quad (23)$$

In particular, because \mathcal{F} is a left adjoint and therefore preserves all existing colimits,

$$\mathcal{F}(\mu) = \coprod_{\mu} \mathcal{F}(1) \quad (24)$$

i.e. $\mathcal{F}(\mu)$ is the coproduct in \mathbf{V} of μ copies of $\mathcal{F}(1)$.

To specialise the general framework of section 2 to varieties of algebras, we stipulate that⁵:

- \mathbf{T} is the opposite of the full subcategory of \mathbf{V} whose objects are the free \mathbf{V} -algebras $\mathcal{F}(\mu)$, as μ ranges over all cardinals.
- \mathbf{S} is the category \mathbf{Set} .
- \triangle is the \mathbf{T} -object $\mathcal{F}(1)$.

It remains to provide an instantiation for the functor $\mathcal{J} : \mathbf{T} \rightarrow \mathbf{S}$. To this end notice that any algebra A yields a functor

$$\mathcal{J}_A : \mathbf{T} \rightarrow \mathbf{Set}$$

that preserves arbitrary products, in the spirit of the Lawvere-Linton functorial semantics of algebraic theories [29, 31, 39, 22, 32, 1, 2]; henceforth we shall write just \mathcal{J} for \mathcal{J}_A . To define \mathcal{J} on objects, we set

$$\mathcal{J}(\mathcal{F}(\mu)) := \mathcal{U}(A)^\mu \quad (25)$$

⁵Notice that this framework is a particular case of that developed in section 4.3, obtained by taking as \mathbf{T} the algebraic theory axiomatising the variety \mathbf{V} and taking \mathbf{T} equal to the full subcategory of the syntactic category of \mathbf{T} , whose objects are powers of the object $\{x.\mathbf{T}\}$.

for any μ . Given a homomorphism $\mathcal{F}(\mu) \rightarrow \mathcal{F}(\nu)$, we construct a function $\mathcal{U}(A)^\nu \rightarrow \mathcal{U}(A)^\mu$ as follows. First, by (24), it suffices to consider the case that $\mu = 1$, i.e. the free algebra be singly generated. Thus, let

$$p: \mathcal{F}(1) \rightarrow \mathcal{F}(\nu) \quad (26)$$

be given. Given an element of $\mathcal{U}(A)^\nu$, i.e. a function

$$a_\nu: \nu \rightarrow \mathcal{U}(A), \quad (27)$$

by the adjunction (22) there is a unique \mathbf{V} -arrow

$$\widehat{a_\nu}: \mathcal{F}(\nu) \rightarrow A. \quad (28)$$

We then have the composition

$$\mathcal{F}(1) \xrightarrow{p} \mathcal{F}(\nu) \xrightarrow{\widehat{a_\nu}} A \quad (29)$$

of (26) and (28). Applying again the adjunction (22) to (29) we obtain an arrow in **Set**

$$\text{ev}(p, a_\nu) := 1 \rightarrow \mathcal{U}(A), \quad (30)$$

i.e. an element of $\mathcal{U}(A)$, called the *evaluation of p at a_ν* . Keeping p fixed and letting a_ν range over all elements (27) of $\mathcal{U}(A)^\nu$, we thus obtain the *evaluation map*

$$\text{ev}(p, -): \mathcal{U}(A)^\nu \rightarrow \mathcal{U}(A). \quad (31)$$

We set

$$\mathcal{J}(p) := \text{ev}(p, -), \quad (32)$$

and this completes the definition of the functor $\mathcal{J}: \mathbf{T} \rightarrow \mathbf{Set}$.

Definition 5.1. A function $\mathcal{U}(A)^\nu \rightarrow \mathcal{U}(A)^\mu$ is called *definable (in the language of \mathbf{V})* if it is in the range of \mathcal{J} , as defined above. In other words, the definable functions $\mathcal{U}(A)^\nu \rightarrow \mathcal{U}(A)^\mu$ are precisely those that can be obtained by evaluating a μ -tuple of elements of $\mathcal{U}(\mathcal{F}(\nu))$ at the ν -tuples of elements of $\mathcal{U}(A)$.

Remark 5.2. Observe that, in the above, \mathcal{J} preserves all products in \mathbf{T} by construction. Moreover, recall that the forgetful functor $\mathcal{U}: \mathbf{V} \rightarrow \mathbf{Set}$ commutes with products in **Set**, because it is adjoint on the right and hence preserves all existing limits. Stated otherwise, products in varieties are direct products. Hence we have an isomorphism of sets $\mathcal{U}(A^\mu) \cong \mathcal{U}(A)^\mu$. Therefore, the replacement of (25) in the definition of \mathcal{J} by $\mathcal{F}(\mu) \mapsto \mathcal{U}(A^\mu)$ would be immaterial.

Let us now consider the categories **D** and **R** in the present algebraic setting. Specialising the definitions in Subsection 2.1, we see that the **D**-objects are all subsets $S \subseteq \mathcal{U}(A)^\mu$, as μ ranges over all cardinals. The **D**-arrows from $S' \subseteq \mathcal{U}(A)^\nu$ to $S \subseteq \mathcal{U}(A)^\mu$ are the definable functions $\mathcal{U}(A)^\nu \rightarrow \mathcal{U}(A)^\mu$, in the sense of Definition 5.1, that restrict to functions $S' \rightarrow S$. We stress that distinct definable functions $\mathcal{U}(A)^\nu \rightarrow \mathcal{U}(A)^\mu$ are regarded as distinct **D**-arrows even when they yield the same restriction $S' \rightarrow S$.

Concerning the category **R**, let us specialise the definitions in Subsection 2.2. The **R**-objects can be naturally identified with the relations R on $\mathcal{U}(\mathcal{F}(\mu))$, as μ ranges over all cardinals. To see this, observe that an **R**-object is, by definition, a relation R on $\text{hom}_{\mathbf{T}}(\mathcal{F}(\mu), \mathcal{F}(1))$. That is, by our choice of \mathbf{T} , R is a relation on $\text{hom}_{\mathbf{V}}(\mathcal{F}(1), \mathcal{F}(\mu))$. By the adjunction (22), homomorphisms $\mathcal{F}(1) \rightarrow \mathcal{F}(\mu)$ are

in natural bijection with the elements of $\mathcal{U}(\mathcal{F}(\mu))$, so that we can regard R as a relation on $\mathcal{U}(\mathcal{F}(\mu))$. Let us henceforth write

$$(\mathcal{F}(\mu), R) \tag{33}$$

to denote an \mathbf{R} -object. We show next that an \mathbf{R} -arrow

$$(\mathcal{F}(\nu), R') \rightarrow (\mathcal{F}(\mu), R)$$

can be naturally identified with a homomorphism

$$h: \mathcal{F}(\mu) \rightarrow \mathcal{F}(\nu)$$

that preserves R with respect to R' , i.e. satisfies

$$\forall p, q \in \mathcal{U}(\mathcal{F}(\mu)): (p, q) \in R \implies (h(p), h(q)) \in R'. \tag{34}$$

Indeed, the \mathbf{R} -arrow $(\mathcal{F}(\nu), R') \rightarrow (\mathcal{F}(\mu), R)$ is by definition a \mathbf{V} -arrow $h: \mathcal{F}(\mu) \rightarrow \mathcal{F}(\nu)$ such that the function

$$h \circ -: \text{hom}_{\mathbf{V}}(\mathcal{F}(1), \mathcal{F}(\mu)) \longrightarrow \text{hom}_{\mathbf{V}}(\mathcal{F}(1), \mathcal{F}(\nu))$$

satisfies the property

$$(p, q) \in R \implies (h \circ p, h \circ q) \in R' \tag{35}$$

for each pair of homomorphisms $p, q: \mathcal{F}(1) \rightarrow \mathcal{F}(\mu)$. Identifying p and q with elements of $\mathcal{U}(\mathcal{F}(\nu))$ as usual via the adjunction (22), we obtain (34) from (35).

Notation 5.3. For the rest of this paper we follow the standard practice in algebra of omitting the underlying set functor. Thus when we write, for instance, $a \in A$, it is understood that we mean $a \in \mathcal{U}(A)$.

6. THE ALGEBRAIC AFFINE ADJUNCTION

Let us specialise the functors $\mathcal{C}: \mathbf{D} \rightarrow \mathbf{R}$ and $\mathcal{V}: \mathbf{R} \rightarrow \mathbf{D}$ to the algebraic setting of section 5.

Recall that \mathbf{V} is a variety of algebras, A is an arbitrary \mathbf{V} -algebra, \mathbf{T} is the opposite of the full subcategory of \mathbf{V} whose objects are the free \mathbf{V} -algebras $\mathcal{F}(\mu)$, as μ ranges over all cardinals, $\Delta := \mathcal{F}(1)$, and $\mathcal{I}: \mathbf{T} \rightarrow \mathbf{Set}$ is the functor defined in (25–32) above. It is appropriate to recall at this stage the notions of operation and congruence.

Definition 6.1. For ν a cardinal, a \mathbf{V} -operation (or, more simply, an *operation*) of *arity* ν is a \mathbf{V} -homomorphism $t: \mathcal{F}(1) \rightarrow \mathcal{F}(\nu)$. The operation h is *finitary* if ν is finite, and *infinitary* otherwise. An *operation on the \mathbf{V} -algebra A* is a function $f: A^\nu \rightarrow A$ that is definable in the sense of Definition 5.1, that is, such that $h = \mathcal{I}(t) := \text{ev}(t, -)$ for some $t: \mathcal{F}(1) \rightarrow \mathcal{F}(\nu)$.

Remark 6.2. Since homomorphisms $t: \mathcal{F}(1) \rightarrow \mathcal{F}(\nu)$ are naturally identified with elements $t \in \mathcal{F}(\nu)$ via the adjunction (22), the preceding definition agrees with the usual notion of operations as *term-definable functions*; one calls t a *defining term* for the operation in question. By a classical theorem of G. Birkhoff (see e.g. [10, Theorem 10.10]) the free algebra $\mathcal{F}(\nu)$ can indeed be represented as the algebra of *terms* —elements of absolutely free algebras— over the set of variables X_ν , modulo the equivalence relation that identifies two such terms if, and only if, they evaluate to the same element in any \mathbf{V} -algebra. For the infinitary case see [44, Ch. III].

Remark 6.3. When, in the sequel, we say that *homomorphisms commute with operations*, we mean that given any \mathbf{V} -homomorphism $h: A \rightarrow B$, any ν -ary operation $t \in \mathcal{F}(\nu)$, and any element $a := (a_\beta)_{\beta < \nu} \in A^\nu$, we have

$$h(\text{ev}_A(t, a)) = \text{ev}_B(t, (h(a_\beta))_{\beta < \nu}), \quad (36)$$

where $\text{ev}_A(t, -): A^\nu \rightarrow A$ and $\text{ev}_B(t, -): B^\nu \rightarrow B$ are the evaluation maps with respect to A and B . That (36) holds follows by direct inspection of the definitions. It is common to write (36) as

$$h(t(a_\nu)) = t(h(a_\nu)), \quad (37)$$

where the algebras A and B over which t is evaluated are tacitly understood.

Definition 6.4. A *congruence* θ on a \mathbf{V} -algebra A is an equivalence relation on A that is *compatible with* (or *preserved by*) *all operations*, i.e. with all definable maps $f: A^\nu \rightarrow A$, where ν is a cardinal. This means that whenever $x_\nu := (x_\beta)_{\beta < \nu}$, $y_\nu := (y_\beta)_{\beta < \nu}$ are ν -tuples of elements of A ,

$$(x_\beta, y_\beta) \in \theta \text{ for each } \beta < \nu \implies (f(x_\nu), f(y_\nu)) \in \theta. \quad (38)$$

Remark 6.5. With the notation of the preceding definition, upon writing $f = \text{ev}(t, -)$ for some defining term $t \in \mathcal{F}(\nu)$ condition (38) reads

$$(x_\beta, y_\beta) \in \theta \text{ for each } \beta < \nu \implies (\text{ev}(t, x_\nu), \text{ev}(t, y_\nu)) \in \theta. \quad (39)$$

Equivalently, with the convention adopted in (37),

$$(x_\beta, y_\beta) \in \theta \text{ for each } \beta < \nu \implies (t((x_\beta)_{\beta < \nu}), t((y_\beta)_{\beta < \nu})) \in \theta. \quad (40)$$

It is a standard fact, even in the infinitary case, that congruences in the sense of Definition 6.4 coincide with congruences defined in terms of kernel pairs; see [31, p. 33] and [44, Ch. II.5].

The Galois connections (\mathbb{C}, \mathbb{V}) of Subsection 2.3 now specialise as follows. Given a subset $S \subseteq A^\mu$, we have

$$\mathbb{C}(S) = \{(p, q) \in \mathcal{F}(\mu) \mid \forall a \in S: \text{ev}(p, a) = \text{ev}(q, a)\}, \quad (41)$$

where $\text{ev}(p, -): A^\mu \rightarrow A$ is, once more, the evaluation map (31).

Lemma 6.6. *For any cardinal μ , and any subset $S \subseteq A^\mu$, the set $\mathbb{C}(S) \subseteq \mathcal{F}(\mu)^2$ is a congruence relation.*

Proof. It is clear that $\mathbb{C}(S)$ is an equivalence relation. To show it is a congruence, let ν be a cardinal and consider two ν -tuples $(x_\beta)_{\beta < \nu}$, $(y_\beta)_{\beta < \nu}$ of elements of $\mathcal{F}(\mu)$. Since the pairs (x_β, y_β) are in $\mathbb{C}(S)$ for each $\beta < \nu$, we have

$$\forall \beta < \nu: \text{ev}(x_\beta, a) = \text{ev}(y_\beta, a), \quad (42)$$

for all $a \in S$. If $t \in \mathcal{F}(\nu)$ is any ν -ary operation on $\mathcal{F}(\mu)$, applying t to (42) we obtain

$$t(\text{ev}(x_\beta, a))_{\beta < \nu} = t(\text{ev}(y_\beta, a))_{\beta < \nu}, \quad (43)$$

that is, more explicitly,

$$\text{ev}(t, (\text{ev}(x_\beta, a))_{\beta < \nu}) = \text{ev}(t, (\text{ev}(y_\beta, a))_{\beta < \nu}). \quad (44)$$

Directly from the definitions one verifies

$$\text{ev}(t, (\text{ev}(x_\beta, a))_{\beta < \nu}) = \text{ev}(t((x_\beta)_{\beta < \nu}), a), \quad (45)$$

so that from (44–45) we obtain

$$\text{ev}(t((x_\beta)_{\beta < \nu}), a) = \text{ev}(t((y_\beta)_{\beta < \nu}), a),$$

for all $a \in S$, and the proof is complete. \square

Remark 6.7. Every congruence on the free algebra $\mathcal{F}(\mu)$ is \triangle -stable (cf. the proof of Proposition 4.15).

Concerning the operator \mathbb{V} , note first that **Set** obviously satisfies Assumption 1. Given a relation R on $\mathcal{F}(\mu)$, we have

$$\mathbb{V}(R) = \bigcap_{(p,q) \in R} \{a \in \mathcal{J}(\mathcal{F}(\mu)) \mid \text{ev}(p, a) = \text{ev}(q, a)\}. \quad (46)$$

Lemma 2.4 asserts that, for any cardinal μ , any relation R on $\mathcal{F}(\mu)$, and any subset $S \subseteq A^\mu$, we have

$$R \subseteq \mathbb{C}(S) \quad \text{if, and only if,} \quad S \subseteq \mathbb{V}(R).$$

In other words, the functions $\mathbb{V}: 2^{\mathcal{F}(\mu)^2} \rightarrow 2^{A^\mu}$ and $\mathbb{C}: 2^{A^\mu} \rightarrow 2^{\mathcal{F}(\mu)^2}$ yield a contravariant Galois connection between the indicated power sets.

Consider subsets $S' \subseteq A^\nu$, $S \subseteq A^\mu$, with μ and ν cardinals, and a D-arrow $f: S' \subseteq A^\nu \rightarrow S \subseteq A^\mu$, i.e. a definable function $f: A^\nu \rightarrow A^\mu$ that restricts to a function $S' \rightarrow S$. Recall from (25–32) that f is induced by a (uniquely determined) homomorphism $h: \mathcal{F}(\mu) \rightarrow \mathcal{F}(\nu)$ via evaluation. We have

$$\mathcal{C}(S) = (\mathcal{F}(\mu), \mathbb{C}(S)) \quad (47)$$

with $\mathbb{C}(S)$ as in (41), and similarly for S' . Recall from section 5 that an R-arrow $(\mathcal{F}(\nu), \mathbb{C}(S')) \rightarrow (\mathcal{F}(\mu), \mathbb{C}(S))$ is naturally identified with a homomorphism $\mathcal{F}(\mu) \rightarrow \mathcal{F}(\nu)$ that preserves $\mathbb{C}(S)$ with respect to $\mathbb{C}(S')$ in the sense of (34). Now \mathcal{C} carries the D-arrow f to the unique R-arrow corresponding to the homomorphism $h: \mathcal{F}(\mu) \rightarrow \mathcal{F}(\nu)$, i.e. we have

$$\mathcal{C}(f) = h. \quad (48)$$

Consider, conversely, R-objects $(\mathcal{F}(\nu), R')$ and $(\mathcal{F}(\mu), R)$, together with an R-arrow $(\mathcal{F}(\nu), R') \rightarrow (\mathcal{F}(\mu), R)$. The latter, by our choice of \mathbb{T} , can be identified with a homomorphism $h: \mathcal{F}(\mu) \rightarrow \mathcal{F}(\nu)$ that preserves R with respect to R' . We have

$$\mathcal{V}(\mathcal{F}(\mu), R) = \mathbb{V}(R) \subseteq \mathcal{J}(\mathcal{F}(\mu)) \quad (49)$$

with $\mathbb{V}(R)$ as in (46), and similarly for $(\mathcal{F}(\nu), R')$. Via evaluation, h induces a definable function $f: A^\nu \rightarrow A^\mu$ that restricts to a function $S' \rightarrow S$, and thus yields a D-arrow $f: S' \subseteq A^\nu \rightarrow S \subseteq A^\mu$; i.e., we have

$$\mathcal{V}(h) = f.$$

The weak affine adjunction (Theorem 2.8) applies to show $\mathcal{C} \dashv \mathcal{V}$.

We shall carry the adjunction $\mathcal{C} \dashv \mathcal{V}$ through to the quotient categories \mathbf{D}^q and \mathbf{R}^q in the algebraic setting.

6.1. The quotient D^q : Affine subsets. Specialising Subsection 3, the quotient category D^q has the same objects as D , namely, all subsets $S \subseteq A^\mu$, as μ ranges over all cardinals. The D -arrows from $S' \subseteq A^\nu$ to $S \subseteq A^\mu$ are the definable functions $A^\nu \rightarrow A^\mu$, in the sense of Definition 5.1, that restrict to functions $S' \rightarrow S$, up to the equivalence relation that identifies two such definable functions if, and only if, they restrict to the same function $S' \rightarrow S$. It is reasonable to call an object $S \subseteq A^\mu$ of D^q an *affine subset (relative to A and V)*.

6.2. The quotient R^q : Presented algebras. Continuing our specialisation of Subsection 3, the quotient category R^q has as objects the pairs $(\mathcal{F}(\mu), R)$, where μ ranges over all cardinals, and R is a relation on (the underlying set of) $\mathcal{F}(\mu)$. The R^q -morphisms $(\mathcal{F}(\nu), R') \rightarrow (\mathcal{F}(\mu), R)$ are the homomorphisms $h: \mathcal{F}(\mu) \rightarrow \mathcal{F}(\nu)$ that preserve R with respect to R' in the sense of (35), up to the equivalence relation that identifies two of them if, and only if, their factorisations through the natural quotient maps $\mathcal{F}(\mu) \twoheadrightarrow \mathcal{F}(\mu)/\overline{R}$ and $\mathcal{F}(\nu) \twoheadrightarrow \mathcal{F}(\nu)/\overline{R'}$ are equal. As already noted, when R and R' are congruences, the factorisations in question are in fact homomorphisms from the algebra $\mathcal{F}(\mu)/R$ to the algebra $\mathcal{F}(\nu)/R'$. We therefore recall a standard

Definition 6.8. We call a pair $(\mathcal{F}(\mu), \theta)$, for μ a cardinal and θ a congruence on $\mathcal{F}(\mu)$, a *presentation (in the variety V)*. We call the algebra $\mathcal{F}(\mu)/\theta$ the *algebra presented by $(\mathcal{F}(\mu), \theta)$* . We write V_p for the category of *presented V -algebras*, having as objects all presentations in V , and as morphisms the V -homomorphisms between the V -algebras presented by them.

Theorem 6.9. *Let V be any (finitely or infinitary) variety of algebras, V_p the associated category of presented V -algebras. Set T to be the opposite of the full subcategory of V whose objects are the free V -algebras $\mathcal{F}(\mu)$, for μ an arbitrary cardinal, $\Delta := \mathcal{F}(1)$, $\mathcal{I}: T \rightarrow \mathbf{Set}$ to be the functor defined in section 5 and R^q be the category defined as in section 3. Then, the category V_p fully embeds into $(R^q)^{\text{op}}$.*

Proof. Consider the functor that sends an object $(\mathcal{F}(\mu), \theta)$ in V_p into the object $(\mathcal{F}(\mu), \theta)$ of $(R^q)^{\text{op}}$. The functor associates with any map $h: (\mathcal{F}(\mu), \theta) \rightarrow (\mathcal{F}(\nu), \theta')$, a map $\bar{h}: \mathcal{F}(\nu) \rightarrow \mathcal{F}(\mu)$ which is the dual of the unique homomorphism extension of the assignment $X_\alpha \mapsto Y_\beta$, where $\{X_\alpha \mid \alpha < \mu\}$ are the free generators of $\mathcal{F}(\mu)$ and Y_β is an arbitrary representative of the θ' -equivalence class $h(X_\alpha/\theta)$. The verification that this is indeed a well-defined functor is straightforward. It remains to prove that it is full and faithful. For the first claim, consider a (representative of the) R^q -arrow $f: (\mathcal{F}(\nu), \theta') \rightarrow (\mathcal{F}(\mu), \theta)$. Since f preserves θ with regard to θ' , the map $h: (\mathcal{F}(\mu), \theta) \rightarrow (\mathcal{F}(\nu), \theta')$ defined by $h(t/\theta) := f(t)/\theta'$ is well-defined and is a homomorphism of V_p -algebras. Now, \bar{h} as defined above, sends a free generator X_α into an arbitrary representative of the θ' -equivalence class $h(X_\alpha/\theta) = f(X_\alpha)/\theta'$, so \bar{h} and f have the same factorisation through the algebras $(\mathcal{F}(\mu), \theta)$ and $(\mathcal{F}(\nu), \theta')$, hence they are the same arrow in R^q . To prove that the functor is faithful, notice that if two arrows $h_1, h_2: (\mathcal{F}(\mu), \theta) \rightarrow (\mathcal{F}(\nu), \theta')$ in V_p are different, then \bar{h}_1 and \bar{h}_2 are different in R and they belong to different equivalence classes in R^q as they factor differently through the quotients. \square

Remark 6.10. While V_p is clearly not a variety of algebras—it is not closed, for example, under isomorphisms—it is equivalent to V . Indeed, we have a functor that sends each presented algebra $(\mathcal{F}(\mu), \theta)$ into the quotient $\mathcal{F}(\mu)/\theta$ in V and acts

identically on maps. It is an exercise to see that such a functor is full, faithful, and dense, hence provides an equivalence of categories.

6.3. Algebraic affine adjunction. Recall the notion of \mathcal{J} -coseparator from Definition 3.5. In the algebraic setting, Assumption 2 always holds:

Lemma 6.11. *The object $\triangle = \mathcal{F}(1)$ is an \mathcal{J} -coseparator for the functor \mathcal{J} defined in (25–32) above.*

Proof. We need to show that, for any cardinal μ , the family of definable functions $f: A^\mu \rightarrow A$ is jointly monic in **Set**. That is, given any two functions $h_1, h_2: S \rightarrow A^\mu$, if $f \circ h_1 = f \circ h_2$ for all definable f , then $h_1 = h_2$. Note that the canonical projection functions $\pi_\alpha: A^\mu \rightarrow A$ of the product A^μ , for $\alpha < \mu$ an ordinal, are definable. Indeed, inspection of the definition of \mathcal{J} shows that the unique homomorphism $\iota_\alpha: \mathcal{F}(1) \rightarrow \mathcal{F}(\mu)$ induced by $X_1 \mapsto X_\alpha$ is such that $\mathcal{J}(\iota_\alpha) = \pi_\alpha$. If now $h_1 \neq h_2$, by the universal property of products there $\alpha < \mu$ with $\pi_\alpha \circ h_1 \neq \pi_\alpha \circ h_2$, as was to be shown. \square

Remark 6.12. In the light of Lemma 6.6 and Theorem 6.9, the image of \mathcal{C}^q ranges within the full subcategory \mathbf{V}_p of \mathbf{R}^q . Thus, without loss of generality, we restrict our attention to this subcategory rather than the whole \mathbf{R}^q .

Specialising Definition 3.7, we obtain functors $\mathcal{C}^q: \mathbf{D}^q \rightarrow \mathbf{V}_p^{\text{op}}$ and $\mathcal{V}^q: \mathbf{V}_p^{\text{op}} \rightarrow \mathbf{D}^q$. As an immediate consequence of Theorem 3.8, Lemma 6.11, Remark 6.12, and Theorem 6.9, we have:

Corollary 6.13 (Algebraic affine adjunction). *Consider any (finitary or infinitary) variety \mathbf{V} of algebras, and fix any \mathbf{V} -algebra A . Let \mathbf{V}_p be the associated category of presented algebras as in Definition 6.8. Let \mathbf{D}^q be the category of affine subsets relative to A and \mathbf{V} , as in Subsection 6.1. The functors $\mathcal{C}^q: \mathbf{D}^q \rightarrow \mathbf{V}_p^{\text{op}}$ and $\mathcal{V}^q: \mathbf{V}_p^{\text{op}} \rightarrow \mathbf{D}^q$ are adjoint with $\mathcal{C}^q \dashv \mathcal{V}^q$.* \square

7. THE ALGEBRAIC Nullstellensatz

Remark 7.1. It is well known that in any (finitary or infinitary) variety \mathbf{V} of algebras we have:

- (1) The monomorphisms are exactly the injective \mathbf{V} -homomorphisms, which we also call embeddings.
- (2) The regular epimorphisms (=the coequalisers of some pair of parallel arrows) are exactly the surjective \mathbf{V} -homomorphisms, which we also call quotient maps.

(See [31, pp. 87–88].) We shall use these basic facts often in this section.

7.1. A Stone-Gelfand-Kolmogorov Lemma. Recall from section 5 that, for a cardinal ν and a given element $a \in A^\nu$, we have the homomorphism

$$\hat{a}: \mathcal{F}(\nu) \rightarrow A. \quad (28)$$

Note that the action of (the underlying function $\mathcal{U}(\hat{a})$ of) (28) is given by

$$p \in \mathcal{F}(\nu) \xrightarrow{\hat{a}} \text{ev}(p, a) \in A. \quad (50)$$

For, applying the adjunction $\mathcal{F} \dashv \mathcal{U}$ to

$$\mathcal{F}(1) \xrightarrow{p} \mathcal{F}(\nu) \xrightarrow{\hat{a}} A \quad (29)$$

we obtain the commutative diagram

$$\begin{array}{ccccc} 1 & \xrightarrow{\check{p}} & \mathcal{U}(\mathcal{F}(\nu)) & \xrightarrow{\mathcal{U}(\hat{a})} & \mathcal{U}(A) \\ & & & \searrow & \\ & & & \text{ev}(p, a) & \end{array}$$

where we write $\check{p}: 1 \rightarrow \mathcal{U}(\mathcal{F}(\nu))$ for the unique function corresponding to $p: \mathcal{F}(1) \rightarrow \mathcal{F}(\nu)$ under the adjunction.

We also have the natural quotient homomorphism

$$q_a: \mathcal{F}(\nu) \twoheadrightarrow \mathcal{F}(\nu)/\mathbb{C}(\{a\}). \quad (51)$$

By construction, q_a preserves the relation $\mathbb{C}(\{a\})$ on $\mathcal{F}(\nu)$ with respect to the identity relation on $\mathcal{F}(\nu)/\mathbb{C}(\{a\})$. Now, \hat{a} preserves the relation $\mathbb{C}(\{a\})$ on $\mathcal{F}(\nu)$ with respect to the identity relation on A . Indeed, if $(p, q) \in \mathbb{C}(\{a\})$ then, by definition, $\text{ev}(p, a) = \text{ev}(q, a)$, whence $\hat{a}(p) = \hat{a}(q)$ by (50). Therefore, by the universal property of the quotient homomorphism there exists a unique homomorphism

$$\gamma_a: \mathcal{F}(\nu)/\mathbb{C}(\{a\}) \longrightarrow A \quad (52)$$

that makes the diagram in Fig. 3 commute.

$$\begin{array}{ccc} \mathcal{F}(\nu) & & \\ \downarrow \hat{a} & \searrow q_a & \\ A & \xleftarrow[\gamma_a]{!} & \mathcal{F}(\nu)/\mathbb{C}(\{a\}) \end{array}$$

FIGURE 3. The Gelfand evaluation γ_a .

Definition 7.2 (Gelfand evaluation). Given a cardinal ν and an element $a \in A^\nu$, the homomorphism (52) above is called the *Gelfand evaluation* (of $\mathcal{F}(\nu)$ at a).

Lemma 7.3 (Stone-Gelfand-Kolmogorov Lemma). *Fix a cardinal ν .*

- (i) *For each $a \in A^\nu$, the Gelfand evaluation γ_a is a monomorphism, and hence its underlying function $\mathcal{U}(\gamma_a)$ is injective.*
- (ii) *Conversely, for each congruence relation θ on $\mathcal{F}(\nu)$, and each homomorphism $e: \mathcal{F}(\nu)/\theta \rightarrow A$, consider the commutative diagram*

$$\begin{array}{ccc} \mathcal{F}(\nu) & & \\ \downarrow e \circ q_\theta & \searrow q_\theta & \\ A & \xleftarrow[e]{} & \mathcal{F}(\nu)/\theta \end{array}$$

where q_θ is the natural quotient homomorphism. Set $a := (e \circ q_\theta(X_\beta))_{\beta < \nu} \in A^\nu$. If e is a monomorphism, then $\theta = \mathbb{C}(\{a\})$, and the commutative diagram above coincides with the one in Fig. 3. (That is, $q_\theta = q_a$, $e = \gamma_a$, and $e \circ q_\theta = \hat{a}$.)

Proof. (i) It suffices to check that the underlying function of γ_a is injective, cf. Remark 7.1. Pick $p, q \in \mathcal{F}(\nu)$ such that $(p, q) \notin \mathbb{C}(\{a\})$. Then, by definition, $\text{ev}(p, a) \neq \text{ev}(q, a)$, and therefore $\widehat{a}(p) \neq \widehat{a}(q)$ by (50). But then, by the definition of Gelfand evaluation, it follows that $\gamma_a(p) \neq \gamma_a(q)$.

(ii) Since e is monic, we have $\ker(e \circ q_\theta) = \ker q_\theta = \theta$. Explicitly,

$$\forall s, t \in \mathcal{F}(\nu) : (s, t) \in \theta \iff e(q_\theta(s)) = e(q_\theta(t)). \quad (53)$$

Since homomorphisms commute with operations, cf. Remark 6.3, and recalling the definition of a , (53) yields

$$\forall s, t \in \mathcal{F}(\nu) : (s, t) \in \theta \iff \text{ev}(s, a) = \text{ev}(t, a). \quad (54)$$

Therefore, by (54), we have $a \in \mathbb{V}(\theta)$. By the Galois connection (7) this is equivalent to

$$\theta \subseteq \mathbb{C}(\{a\}). \quad (55)$$

For the converse inclusion, if $(u, v) \in \mathbb{C}(\{a\})$, then $\text{ev}(u, a) = \text{ev}(v, a)$, and therefore $(u, v) \in \theta$ by (54). This proves $\theta = \mathbb{C}(\{a\})$, and therefore $q_\theta = q_a$. To show $\widehat{a} = e \circ q_a$, note that, by the definition of \widehat{a} and the universal property of $\mathcal{F}(\nu)$, they both are the (unique) extension of the function $X_\beta \mapsto \text{ev}(X_\beta, (e \circ q_\theta(X_\beta)))$, for $\beta < \nu$. \square

7.2. Transforms. For a congruence relation θ on $\mathcal{F}(\nu)$, we now consider the natural quotient homomorphism

$$q_\theta : \mathcal{F}(\nu) \rightarrow \mathcal{F}(\nu)/\theta, \quad (56)$$

together with the product $\prod_{a \in \mathbb{V}(\theta)} \mathcal{F}(\nu)/\mathbb{C}(\{a\})$ and its projections

$$\pi_a : \prod_{a \in \mathbb{V}(\theta)} \frac{\mathcal{F}(\nu)}{\mathbb{C}(\{a\})} \rightarrow \frac{\mathcal{F}(\nu)}{\mathbb{C}(\{a\})}. \quad (57)$$

We also consider the power $A^{\mathbb{V}(\theta)}$ and its projections

$$p_a : A^{\mathbb{V}(\theta)} \rightarrow A. \quad (58)$$

The morphisms (52–58) yield the commutative diagrams—one for each $a \in \mathbb{V}(\theta)$ —in Fig. 4, where σ_θ and ι_θ are the unique homomorphisms whose existence is granted

$$\begin{array}{ccccc}
 & & \gamma_\theta := \iota_\theta \circ \sigma_\theta & & \\
 & & \downarrow ! & & \\
 \mathcal{F}(\nu)/\theta & \xrightarrow[\downarrow !]{\sigma_\theta} & \prod_{a \in \mathbb{V}(\theta)} \frac{\mathcal{F}(\nu)}{\mathbb{C}(\{a\})} & \xrightarrow[\downarrow !]{\iota_\theta} & A^{\mathbb{V}(\theta)} \\
 \downarrow q & \swarrow \pi_a & \downarrow \gamma_a \circ \pi_a & \searrow p_a & \\
 \mathcal{F}(\nu)/\mathbb{C}(\{a\}) & \xrightarrow{\gamma_a} & A & &
 \end{array}$$

FIGURE 4. The Gelfand and Birkhoff transforms γ_θ and σ_θ .

by the universal property of the products $\prod_{a \in \mathbb{V}(\theta)} \frac{\mathcal{F}(\nu)}{\mathbb{C}(\{a\})}$ and $A^{\mathbb{V}(\theta)}$, respectively.

Definition 7.4 (Gelfand and Birkhoff transforms). Given a cardinal ν and a congruence θ on $\mathcal{F}(\nu)$, the homomorphisms $\gamma_\theta := \iota_\theta \circ \sigma_\theta$ and σ_θ given by the commutative diagram above are called the *Gelfand* and the *Birkhoff transforms* (of $\mathcal{F}(\nu)/\theta$ with respect to A), respectively.

Lemma 7.5. *With the notation above, and for each $a \in A$, the homomorphisms $\pi_a \circ \sigma_\theta$ and ι_θ are surjective and injective, respectively.*

Proof. It is clear that $\pi_a \circ \sigma_\theta$ is onto, because $q: \mathcal{F}(\nu)/\theta \rightarrow \mathcal{F}(\nu)/\mathbb{C}(\{a\})$ is onto (cf. Remark 7.1). Concerning ι_θ , let $x, y \in \prod_{a \in \mathbb{V}(A)} \mathcal{F}(\nu)/\mathbb{C}(\{a\})$, and suppose $\iota_\theta(x) = \iota_\theta(y)$. With reference to the commutative diagram in Fig. 4, for each $a \in \mathbb{V}(\theta)$ we have $p_a(\iota_\theta(x)) = p_a(\iota_\theta(y))$, and therefore $\gamma_a(\pi_a(x)) = \gamma_a(\pi_a(y))$. Since γ_a is a monomorphism for each a by Lemma 7.5, we infer $\pi_a(x) = \pi_a(y)$ for each a , and hence $x = y$ by the universal property of the product $\prod_{a \in \mathbb{V}(A)} \mathcal{F}(\nu)/\mathbb{C}(\{a\})$. \square

7.3. The algebraic Nullstellensatz.

Definition 7.6 (Radical). For a cardinal ν and a relation R on $\mathcal{F}(\nu)$, we call the congruence

$$\bigcap_{a \in \mathbb{V}(R)} \mathbb{C}(\{a\})$$

the *radical of R (with respect to the \mathbb{V} -algebra A)*. A congruence θ on $\mathcal{F}(\nu)$ is *radical (with respect to A)* if $\theta = \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(\{a\})$.

Note that the inclusion

$$\theta \subseteq \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(\{a\}), \quad (59)$$

always holds, cf. (9).

Theorem 7.7 (Algebraic Nullstellensatz). *For any \mathbb{V} -algebra A , any cardinal ν , and any congruence θ on $\mathcal{F}(\nu)$. The following are equivalent.*

- (i) $\mathbb{C}(\mathbb{V}(\theta)) = \theta$.
- (ii) $\theta = \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(\{a\})$, i.e. θ is a radical congruence with respect to A .
- (iii) The Birkhoff transform $\sigma_\theta: \mathcal{F}(\nu)/\theta \rightarrow \prod_{a \in \mathbb{V}(\theta)} \frac{\mathcal{F}(\nu)}{\mathbb{C}(\{a\})}$ is a subdirect embedding.

Remark 7.8. In the proof that follows we apply three standard results in universal algebra, namely, [10, Theorems 7.15, 6.15, and 6.20]. Although in [10] these results are stated and proved for finitary varieties, the same proofs work for infinitary ones.

Proof. The hypotheses of Theorem 2.5 are satisfied: the terminal object in **Set** is a singleton $\{a\}$, and the family of functions $\{a\} \rightarrow \mathbb{V}(R)$ —i.e. the elements of $\mathbb{V}(R)$ — is obviously jointly epic. This proves the equivalence of (i) and (ii).

(ii) \Leftrightarrow (iii). By [10, Theorem 7.15], given any algebra B and a family $\{\theta_i\}_{i \in I}$ of congruences on B , the natural homomorphism $h: B \rightarrow \prod_{i \in I} B/\theta_i$ induced by the quotient homomorphisms $q_{\theta_i}: B \rightarrow B/\theta_i$ is an embedding if, and only if, $\bigcap_{i \in I} \theta_i$ is the identity congruence Δ on B . Taking

$$B := \mathcal{F}(\nu)/\theta \quad \text{and} \quad \{\theta_i\} := \{\mathbb{C}(\{a\})\}_{a \in \mathbb{V}(\theta)},$$

we obtain the natural homomorphism

$$h: \mathcal{F}(\nu)/\theta \longrightarrow \prod_{a \in \mathbb{V}(\theta)} \frac{\mathcal{F}(\nu)/\theta}{\mathbb{C}(\{a\})/\theta}, \quad (60)$$

where $\mathbb{C}(\{a\})/\theta$ denotes the set $\{(p/\theta, q/\theta) \in \mathcal{F}(\nu)/\theta \mid (p, q) \in \mathbb{C}(\{a\})\}$, which is easily seen to be a congruence relation on $\mathcal{F}(\nu)/\theta$. It is clear by construction that if h is an embedding, then it is subdirect. Hence we have:

$$h \text{ is a subdirect embedding} \iff \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(\{a\})/\theta = \Delta/\theta \quad (61)$$

For each $a \in \mathbb{V}(\theta)$, by the Galois connection Lemma 2.4 we have $\theta \subseteq \mathbb{C}(\{a\})$. Therefore, by the Second Isomorphism Theorem [10, Theorem 6.15],

$$\forall a \in \mathbb{V}(\theta) : \frac{\mathcal{F}(\nu)/\theta}{\mathbb{C}(\{a\})/\theta} \cong \mathcal{F}(\nu)/\mathbb{C}(\{a\}). \quad (62)$$

From (61–62) we see:

$$h \text{ is a subdirect embedding} \iff \sigma_\theta \text{ is a subdirect embedding.} \quad (63)$$

Finally, upon recalling that, by [10, Theorem 6.20], the mapping $\theta' \mapsto \theta'/\theta$ is an isomorphism of lattices between the lattice of congruences of $\mathcal{F}(\nu)$ extending θ and the lattice of congruences of $\mathcal{F}(\nu)/\theta$, we have

$$\bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(\{a\})/\theta = \Delta/\theta \iff \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(\{a\}) = \theta. \quad (64)$$

In conclusion, (60–64) amount to the equivalence between (ii) and (iii). \square

Remark 7.9. Since Birkhoff’s influential paper [7] the theory of algebras definable by operations of finite arity only has been developed intensively. In [7, Theorem 1] Birkhoff pointed out, by way of motivation for his main result, that the Lasker-Noether theorem [38] generalises easily to algebras whose congruences satisfy the ascending chain condition, even in the presence of operations of infinite arity. His main result [7, Theorem 2] then showed how to extend the Lasker-Noether theorem to any variety of algebras, without any chain condition, provided however that all operations be finitary. In short, Birkhoff’s Subdirect Representation theorem (see e.g. [26, Theorem 2.6] for a textbook treatment) fails for infinitary varieties of algebras. Much of the remaining general theory, however, carries over to the infinitary case. The two classical references on infinitary varieties are [44, 31]. Linton’s paper [31], in particular, extended Lawvere’s categorical treatment of universal algebra [29, 30].

As a consequence of Theorem 7.7, the adjunction $\mathcal{C}^q \dashv \mathcal{V}^q$ need not be a duality for the whole variety \mathbf{V} . The case of MV-algebras, treated in [34], is a non-trivial example of the adjunction $\mathcal{C}^q \dashv \mathcal{V}^q$ that only fixes a subclass of the variety \mathbf{V} . The following corollary provides some sufficient and some necessary conditions for the whole variety \mathbf{V} to be fixed under the composition $\mathcal{C}^q \circ \mathcal{V}^q$; it will also be useful to prove the dualities for Boolean algebras and C^* -algebras in sections 10 and 11 of Part 4. In the light of Remark 6.10, instead than \mathbf{V}_p we work with the equivalent category \mathbf{V} .

- Corollary 7.10.** (1) Let \mathbf{V} be a semisimple variety and suppose there is a cardinal κ such that the number of pairwise non-isomorphic simple algebras in \mathbf{V} is less than κ . Let A be the coproduct of all pairwise non-isomorphic simple algebras in \mathbf{V} , then the composition $\mathcal{C}^q \circ \mathcal{V}^q$ fixes all algebras in \mathbf{V} .
- (2) Let \mathbf{V} be a finitary variety and suppose there is a cardinal κ such that the number of pairwise non-isomorphic subdirectly irreducible algebras in \mathbf{V} is less than κ . Let A be the coproduct of all pairwise non-isomorphic subdirectly irreducible algebras in \mathbf{V} , then the composition $\mathcal{C}^q \circ \mathcal{V}^q$ fixes all algebras in \mathbf{V} .
- (3) Let \mathbf{V} be a finitary variety. Suppose that functors \mathcal{C}^q and \mathcal{V}^q have been defined relatively to some arbitrary algebra A in \mathbf{V} . The algebra A contains (up to isomorphism) all subdirectly irreducible algebras in the variety if, and only if, the composition $\mathcal{C}^q \circ \mathcal{V}^q$ fixes the whole variety \mathbf{V} .
- (4) Let \mathbf{V} be a finitary variety and suppose that the composition $\mathcal{C}^q \circ \mathcal{V}^q$ fixes all algebras in \mathbf{V} , then A generates the variety \mathbf{V} .

Proof. We prove item (1). We set the algebra A to be the aforementioned coproduct. Notice that, if an algebra $\mathcal{F}(\mu)/\psi$ is simple algebra, then it canonically embeds into A , for A is the coproduct of all pairwise non isomorphic simple algebras. So, by Lemma 7.3, $\psi = \mathbb{C}(\{a\})$ for some $a \in \mathbb{V}(\psi)$. Let now $\mathcal{F}(\mu)/\theta$ be any algebra in \mathbf{V} . Since the variety \mathbf{V} is semisimple, $\mathcal{F}(\mu)/\theta$ is semisimple, so by definition there is a subdirect embedding of $\mathcal{F}(\mu)/\theta$ into simple algebras. Each of the simple algebras in \mathbf{V} is isomorphic to $\mathcal{F}(\mu)/\mathbb{C}(\{a\})$ for some cardinal μ and some $a \in A^\mu$. Since the decomposition is subdirect $\theta \subseteq \mathbb{C}(\{a\})$, so by (7) we have $a \in \mathbb{V}(\theta)$. But as seen in Figure 4, there is only one arrow from $\mathcal{F}(\mu)/\theta$ into $\prod_{a \in \mathbb{V}(\{a\})} \mathcal{F}(\mu)/\mathbb{C}(\{a\})$ and this is the Birkhoff transform. Thus, Theorem 7.7 can be applied yielding that the algebra $\mathcal{F}(\mu)/\theta$ is fixed by $\mathbb{C}^q \circ \mathbb{V}^q$ and a straightforward verification of the definitions of \mathcal{C}^q and \mathcal{V}^q give us $\mathcal{C}^q(\mathcal{V}^q((\mathcal{F}(\mu), \theta))) \cong (\mathcal{F}(\mu), \theta)$.

For the proof of item (2), replace “simple” for “subdirectly irreducible” in the proof of item (1). The proof then goes through, upon noticing that by Birkhoff theorem in a finitary variety any algebra is the subdirect product of subdirectly irreducible algebras.

For the proof of item (3), the sufficiency is again obtained as in (2). To see that also the other direction holds, notice that a congruence of $\mathcal{F}(\mu)$ presents a subdirectly irreducible algebra if, and only if, it is *completely meet irreducible* in the lattice of congruences of $\mathcal{F}(\mu)$ (see e.g. [36, Lemma 4.43]). Recall that an element x of a lattice L is completely meet irreducible if whenever $x = \bigvee K$ for some subset K of L , then x must belong to K . Now, suppose that the composition $\mathcal{C} \circ \mathbb{V}$ fixes all algebras in \mathbf{V} , let $\mathcal{F}(\mu)/\theta$ be a subdirectly irreducible algebra. In particular we have $\mathbb{C} \circ \mathbb{V}(\theta) = \theta$, so by Theorem 7.7 $\theta = \bigcap_{a \in \mathbb{V}(\theta)} \mathbb{C}(\{a\})$. But θ is completely meet irreducible, so there exists $a \in \mathbb{V}(\theta)$ such that $\theta = \mathbb{C}(\{a\})$. By Lemma 7.3 this entails that $\mathcal{F}(\mu)/\theta$ embeds into A and the claim is proved.

Finally, to prove item (4), notice that, by (3) A must contain all subdirectly irreducible algebras in \mathbf{V} , hence it generates the variety. \square

8. THE TOPOLOGICAL Nullstellensatz

Having settled the characterisation of fixed points on the algebraic side, we turn to the study of the fixed points on the geometric side of the adjunction. Unfortunately, we are not able at this stage of giving a characterisation as satisfactory

as the one for the algebraic side. Nonetheless, in this section we collect some general facts that will be useful to obtain Stone and Gelfand dualities in the next sections. We shall assume henceforth that the operator $\mathbb{V} \circ \mathbb{C}$ is topological⁶ i.e., $\mathbb{V} \circ \mathbb{C}(X \cup Y) = \mathbb{V} \circ \mathbb{C}(X) \cup \mathbb{V} \circ \mathbb{C}(Y)$. We shall topologise the set A by declaring closed the sets of the form $\mathbb{V} \circ \mathbb{C}(S)$ for some $S \subseteq A$. We call this the *Zariski* topology or the $\mathbb{V} \circ \mathbb{C}$ topology.

For any cardinal μ the power A^μ can be endowed with at least two natural topologies:

- (1) The product topology w.r.t. the $\mathbb{V} \circ \mathbb{C}$ topology on A .
- (2) The $\mathbb{V} \circ \mathbb{C}$ topology given by definable definable functions from A^μ into A i.e., where the closed subsets are of the form

$$\mathbb{V}(\theta) := \{s \in A^\mu \mid p(s) = q(s) \quad \forall (p, q) \in \theta\}$$

for $\theta \subseteq \mathcal{F}^2(\mu)$.

We are interested in cases in which the two topologies above coincide on A^μ for any cardinal μ .

Definition 8.1. We say that a function $f: A^\mu \rightarrow A$ is *strongly continuous* if the pre-images of a sets of the form $\mathbb{V} \circ \mathbb{C}(S)$ with $S \subseteq A$ can be written as $\mathbb{V} \circ \mathbb{C}(T)$ with $T \subseteq A^\mu$.

Strong continuity generalises spectral maps in the setting of Stone duality for distributive lattices. Notice that strong continuity implies continuity.

Lemma 8.2. *Definable functions are strongly continuous (hence in particular continuous) with respect to the $\mathbb{V} \circ \mathbb{C}$ topology.*

Proof. Let f be a definable function from A^μ into A , with definable term $\lambda((X)_{\alpha < \mu})$ and let $C = \mathbb{V} \circ \mathbb{C}(S)$ for some $S \subseteq A$. Consider the set $\theta = \{(s(\lambda), t(\lambda)) \mid (s, t) \in \mathbb{C}(S)\}$, we claim that $f^{-1}[C] = \mathbb{V}(\theta)$. Indeed $d \in f^{-1}[C]$ if, and only if, $\exists c \in C$ such that $f(d) = c$ if, and only if, $\exists c \forall (s, t) \in \mathbb{C}(S), s(f(c)) = t(f(c))$, if and only if, $c \in \mathbb{V}(\theta)$. \square

As an immediate consequence of Lemma 8.2 and the fact that projections are definable functions, we observe that the product topology is coarser than the $\mathbb{V} \circ \mathbb{C}$ topology.

Lemma 8.3 (Co-Nullstellensatz). *Assume that the $\mathbb{V} \circ \mathbb{C}$ topology on A is Hausdorff and that all definable functions are continuous with respect to the product topology. Then a set $S \subseteq A^\mu$ is closed in the product topology if, and only if, $\mathbb{V}(\mathbb{C}(S)) = S$.*

Proof. Let us write \overline{S} for the smallest closed set in the product topology that contains S . As noticed above, the product topology is coarser than the $\mathbb{V} \circ \mathbb{C}$ topology, so we have $\mathbb{V} \circ \mathbb{C}(S) \subseteq \overline{S}$.

To prove the other direction, notice that if X is any topological space, and Y is Hausdorff, then for any two continuous functions $f, g: X \rightarrow Y$ the solution set of the equation $f = g$ is a closed subset of X , [21, 1.5.4]. Now, by assumption A is Hausdorff and definable functions are continuous by Lemma 8.2, so for any pair of terms (s, t) , the set $\mathbb{V}(s, t)$ is closed in product topology. On the other hand,

⁶This is an actual restriction, as the condition may fail in an arbitrary variety. However it holds for all classical dualities mentioned in this paper.

$\mathbb{V}(R) = \mathbb{V}(\bigcup_{(s,t) \in R} \{(s,t)\}) = \bigcap_{(s,t) \in R} \mathbb{V}(s,t)$ holds by Lemma 2.4. We conclude that $\mathbb{V}(R)$ is closed in the product topology for any subset R of $\mathcal{F}_\mu \times \mathcal{F}_\mu$. Thus we obtain the inclusion $\overline{S} \subseteq \mathbb{V}(\mathbb{C}(S))$. \square

Corollary 8.4. *Suppose \mathbb{V} is finitary. If the $\mathbb{V} \circ \mathbb{C}$ topology on A is discrete, then the $\mathbb{V} \circ \mathbb{C}$ topology and the product topology coincide.*

Proof. If the $\mathbb{V} \circ \mathbb{C}$ topology on A is discrete then it obviously is Hausdorff. In addition all finite products are also discrete, and this can be shown to imply that definable functions are continuous with respect to the product topology on A^μ for any cardinal μ , because the variety is finitary. Thus the assumptions of Lemma 8.3 are met and the corollary follows. \square

Part 4. Three classical examples and one epilogue

9. THE CLASSICAL AFFINE ADJUNCTION

Continuing the notation in the Introduction, we consider an algebraically closed field k , and finitely many variables $X := \{X_1, \dots, X_n\}$, $n \geq 0$ an integer. Then k -algebras and their homomorphisms form a finitary variety in the sense of Birkhoff. The k -algebra freely generated by X is the polynomial ring $k[X]$. Congruences on any k -algebra are in one-one inclusion-preserving correspondence with ideals. We shall now apply the results of Part 3 to derive a form of the *Nullstellensatz*, with the *proviso* that congruences are conveniently represented by ideals. We let \mathbb{V} be the variety of k -algebras, and we let $A := k$. The details then depend on what definition one takes for the notion of radical ideal. We shall use:

Definition 9.1. An ideal of a k -algebra is *radical* if, and only if, it is an intersection of maximal ideals.

We shall need a classical result from commutative algebra; see e.g. [3].

Lemma 9.2 (Zariski's Lemma). *Let F be any field, and suppose E is a finitely generated F -algebra that is itself a field. Then E is a finite field extension of F .* \square

Specialising the Stone-Gelfand-Kolmogorov Lemma 7.3 to the ring-theoretic setting now yields:

Lemma 9.3 (Ring-theoretic Stone-Gelfand-Kolmogorov). *An ideal I of $k[X]$ is maximal if, and only if, there exists $a \in k^n$ such that $I = \mathbb{C}(\{a\})$.*

Proof. Assume $I = \mathbb{C}(\{a\})$, and consider the Gelfand evaluation $\gamma_a: k[X]/\mathbb{C}(\{a\}) \rightarrow k$ of Definition 7.2. By Lemma 7.3, γ_a is an embedding. From the fact that γ_a is a homomorphism of k -algebras it follows at once that it is onto k , and hence an isomorphism. Moreover k , being a field, is evidently simple in the universal-algebraic sense, i.e. it has no non-trivial ideals. Hence $\mathbb{C}(\{a\})$, the kernel of the homomorphism $q_a: k[X] \rightarrow k[X]/I$ as in (51), is maximal (by direct inspection, or using the more general [10, Theorem 6.20]).

Conversely, assume that I is maximal, and consider the natural quotient map $q_I: k[X] \rightarrow k[X]/I$. Then $k[X]/I$ is a simple finitely generated k -algebra, and hence a field. By Zariski's Lemma 9.2, $k[X]/I$ is a finite field extension of k ; since k is algebraically closed, k and $k[X]/I$ are isomorphic. Applying Lemma 7.3 with $e: k[X]/I \rightarrow k$ the preceding isomorphism completes the proof. \square

Corollary 9.4 (Ring-theoretic *Nullestellensatz*). *For any ideal I of $k[X]$, the following are equivalent.*

- (i) $\mathbb{C}(\mathbb{V}(I)) = I$.
- (ii) I is radical.

Proof. Immediate consequence of Lemma 9.3 together with Theorem 7.7. \square

It is now possible to functorialise the above along the lines of the first part of this paper, thereby obtaining the usual classical algebraic adjunction. We do not spell out the details.

10. STONE DUALITY FOR BOOLEAN ALGEBRAS

In this section we derive Stone duality for Boolean algebras from the general adjunction. Let \mathbf{V} be the variety of Boolean algebras and their homomorphisms, and set A to be two-element Boolean algebra $\{0, 1\}$. By Corollary 6.13 we have a dual adjunction between \mathbf{V}_p and \mathbf{D}^q given by the functors \mathcal{C}^q and \mathcal{V}^q . We are interested in characterising the fixed points of this adjunction. We begin with the algebraic side. Recall the following:

Lemma 10.1 ([7, Lemma 1]). *To within an isomorphism, the only subdirectly irreducible Boolean algebra is $\{0, 1\}$.*

Corollary 10.2. *With \mathbf{V} and A as in the above, and with reference to the functors of Corollary 6.13, one has*

$$\mathcal{C}^q(\mathcal{V}^q((\mathcal{F}(\mu), R))) \cong (\mathcal{F}(\mu), R)$$

for any $(\mathcal{F}(\mu), R) \in \mathbf{V}_p$.

Proof. Apply Corollary 7.10 item 1 in view of Lemma 10.1. \square

We now turn to the side of affine subsets. The category \mathbf{D}^q is given by subsets of $\{0, 1\}^\mu$ for μ ranging among all cardinals, and definable maps among them. The Zariski ($=\mathbb{V} \circ \mathbb{C}$) topology on $\{0, 1\}$ is discrete as $\{0\} = \mathbb{V}(0, x)$ and $\{1\} = \mathbb{V}(1, x)$.

Lemma 10.3. *Fix a cardinal μ . A set $S \subseteq \{0, 1\}^\mu$ is closed in the product topology if, and only if, $\mathbb{V}(\mathbb{C}(S)) = S$.*

Proof. The topology on A is discrete and Boolean algebras form a finitary variety, so the claim follows from Corollary 8.4. \square

So the space $\{0, 1\}^\mu$ is a *Cantor cube*, i.e. it is topologised by $\mathbb{V} \circ \mathbb{C}$ according to the product topology, $\{0, 1\}$ having the discrete topology.

Corollary 10.4. *Let \mathbf{V} be the variety of Boolean algebras and their homomorphisms, and let A be the Boolean algebra $\{0, 1\}$. With reference to the functors of Corollary 6.13, one has that for any closed set $S \in \mathbf{D}^q$,*

$$\mathcal{V}^q(\mathcal{C}^q(S)) \cong S.$$

Proof. By Lemma 10.3 and direct inspection of the definitions. \square

Corollary 10.5. *The category of Boolean algebras with their homomorphisms is dually equivalent to the category of closed subspaces of the Cantor cubes $\{0, 1\}^\mu$ with continuous maps among them.*

Proof. By Corollary 10.2 the whole category \mathbf{V} is fixed by the composition $\mathcal{C}^q \circ \mathcal{V}^q$. By Corollary 10.4 the full subcategory of closed subsets in \mathbf{D}^q is fixed by the composition $\mathcal{V}^q \circ \mathcal{C}^q$. \square

The last result needed to obtain Stone duality in classical form is an intrinsic characterisation of the closed subspaces of $\{0, 1\}^\kappa$ for κ any cardinal. This is a specific instance of a general problem in abstract topology: given a topological space E , characterise the topological spaces which are subspaces of E^κ . Such spaces are known as E -compact spaces [46, Section 1.4].

Lemma 10.6. *The category of compact, Hausdorff, totally disconnected spaces with continuous maps among them is equivalent to the category \mathbf{D}^q .*

Proof. It is enough to prove that for any compact, Hausdorff, totally disconnected space X , there exists a cardinal μ and closed subset S of $\{0, 1\}^\mu$ such that X is homeomorphic to S . The rest is routine. To prove the claim notice that by [27, Lemma 4.5, pag. 116] given a family F of continuous functions from a Hausdorff space X into spaces Y_f the *evaluation* map $e: X \rightarrow \prod_{f \in F} Y_f$ defined as $e(x)_f := f(x)$ is a homeomorphism between X and $f[X]$, provided that for any $p \in X$ and any closed subset C such that $p \notin C$ there exists $f \in F$ such that $f(p) \notin f[C]$. Given a compact, Hausdorff, totally disconnected space X , we therefore consider the family F of all continuous functions from X to $\{0, 1\}$. If C is a closed subset of X and $p \in X \setminus C$, then there exists a clopen K which extends C and does not contain p . Consider the function

$$f(x) := \begin{cases} 0 & \text{if } x \in K \\ 1 & \text{otherwise.} \end{cases}$$

It is straightforward to see that the function f belongs to F . \square

Corollary 10.7 (Stone 1936). *The category of Boolean algebras with their homomorphisms is dually equivalent to the category of compact, Hausdorff, totally disconnected spaces with continuous maps among them.*

Proof. By composing the equivalences of Corollary 10.5 and the one of Lemma 10.6. \square

11. GELFAND DUALITY FOR C^* -ALGEBRAS

A (*complex, commutative, unital*) C^* -algebra is a complex commutative Banach algebra A (always unital, with identity element written 1) equipped with an involution $*$: $A \rightarrow A$ satisfying $\|xx^*\| = \|x\|^2$ for each $x \in A$. Henceforth, ‘ C^* -algebra’ means ‘complex commutative until C^* -algebra’. The category \mathbf{C}^* has as objects C^* -algebras, and as morphisms their $*$ -homomorphisms, i.e. the complex-algebra homomorphisms preserving the involution and 1. If X is a any compact Hausdorff space, let $\mathbf{C}(X, \mathbf{C})$ denote the complex algebra of all continuous complex-valued functions on X , with operations defined pointwise. Equipped with the involution $*$ given by pointwise conjugation, and with the supremum norm, this is a C^* -algebra. The landmark Gelfand-Naimark Theorem (commutative version) states that, in fact, any C^* -algebra is naturally representable in this manner. A functorial version of the theorem leads to *Gelfand duality*: the category \mathbf{C}^* is dually equivalent to the category of \mathbf{KHaus} of compact Hausdorff spaces and continuous maps.

In this section we show how Gelfand duality fits in the framework of affine adjunctions developed above. The first important fact is that we can work at the level of the algebraic adjunction. For this, we first recall that $x \in A$ is *self-adjoint* if it is fixed by $*$, i.e. if $x^* = x$. Further, we recall that self-adjoint elements carry a partial order which may be defined in several equivalent ways; see e.g. [17, Section 8.3]. For our purposes here it suffices to define a self-adjoint element $x \in A$ to be *non-negative*, written $x \geq 0$, if there exists a self-adjoint $y \in A$ such that $x = y^2$. There is a functor $U: \mathbf{C}^* \rightarrow \mathbf{Set}$ that takes a \mathbf{C}^* -algebra A to the collection of its non-negative self-adjoint elements whose norm does not exceed unity:

$$U(A) := \{x \in A \mid x^* = x, 0 \leq x, \|x\| \leq 1\}.$$

In particular, $U(\mathbf{C}) = [0, 1]$, the real unit interval. In the following we always topologies $[0, 1]$ with its Euclidean topology, and powers $[0, 1]^S$ with the product topology. It is elementary that the restriction of a $*$ -homomorphism $A \rightarrow B$ to $U(A)$ induces a function $U(A) \rightarrow U(B)$, for all \mathbf{C}^* -algebras A and B , so that U is indeed a functor.

Theorem 11.1. *The category \mathbf{C}^* is equivalent to the category \mathbf{V}^* of models of a (necessarily infinitary) algebraic variety. Moreover, under this equivalence the underlying set functor of the variety naturally corresponds to the functor $U: \mathbf{C}^* \rightarrow \mathbf{Set}$ above. The left adjoint F to the functor U associates to a set S the \mathbf{C}^* -algebra of all complex valued, continuous functions on the compact Hausdorff space $[0, 1]^S$.*

Proof. It is well known that the unit-ball functor on \mathbf{C}^* -algebras is monadic over \mathbf{Set} , see [37, Theorem 1.7]. The functor U that we are considering here is a variant of the unit-ball functor. See [40] for further background and results. The fact that no finitary variety can dualise \mathbf{KHaus} was proved, as a consequence of a considerably stronger result, in [4]. Together with Gelfand duality this shows that \mathbf{V}^* cannot be finitary. \square

Remark 11.2. In [25], Isbell proved that there is a finite set of finitary operations, along with a single infinitary operation of countably infinite parity, that generate all operations in \mathbf{V}^* . It has been a long-standing open problem to provide a manageable equational axiomatisation of \mathbf{V}^* . A solution to this problem appears in [33], where a *finite* axiomatisation is provided. The interested reader is referred to [33] for details. For our purposes here, we do not need an explicit presentation of \mathbf{V}^* . Indeed, we shall identify \mathbf{C}^* -algebras with objects of \mathbf{V}^* whenever convenient, it being understood that this identification is via Theorem 11.1.

We start by setting:

- (1) $\mathbf{V} := \mathbf{V}^*$, and
- (2) $A := U(\mathbf{C}) = [0, 1]$.

Corollary 6.13 ensures that there exists a dual adjunction $\mathcal{C}^q \dashv \mathcal{V}^q$ between \mathbf{V}^* and the category of subsets of $[0, 1]^\mu$ —with μ ranging among all cardinals— and definable maps.

The characterisation of the fixed points of the compositions $\mathcal{C}^q \circ \mathcal{V}^q$ and $\mathcal{V}^q \circ \mathcal{C}^q$ is now very similar to the one in Stone duality.

Lemma 11.3.

- (1) *The \mathbf{C}^* -algebra \mathbf{C} is the only simple algebra in \mathbf{V}^* .*
- (2) *The variety \mathbf{V}^* semisimple.*

Proof. The first item amounts to the standard fact that a quotient of a C^* -algebra modulo an ideal I is isomorphic to \mathbf{C} if, and only if, I is maximal. The second item amounts to the equally well-known fact that each C^* algebra has enough maximal ideals, hence it is a subdirect product of copies of \mathbf{C} . \square

Corollary 11.4. *Every commutative C^* -algebra is fixed by the composition $\mathcal{C}^q \circ \mathcal{V}^q$.*

Proof. By combining Proposition 7.10 and Lemma 11.3. \square

We now turn to the characterisation of the fixed points in the geometric side.

Lemma 11.5. *A function f from $S \subseteq [0, 1]^\mu \rightarrow T \subseteq [0, 1]^\nu$ is definable if, and only if, f is continuous with respect to the product topologies, where $[0, 1]$ has the Euclidean topology.*

Proof. For any fixed cardinal μ , by Theorem 11.1 the underlying set of the algebra in \mathbf{V}^* freely generated by a set of cardinality μ is $U(F(\mu))$, that is, the collection of all continuous functions from $[0, 1]^\mu$ to $[0, 1]$. By definition, a function $f: S \subseteq A^\mu \rightarrow T \subseteq A^\nu$ is definable if, and only if, there exists a family of elements $(t_\beta)_{\beta < \nu}$ of elements of $U(F(\mu))$ such that for any $x \in S$, $f(x) = (t(x)_\beta)_{\beta < \nu}$. This proves the lemma. \square

Lemma 11.6. *A subset S of $[0, 1]^\mu$ is closed in the Zariski ($=\mathbb{V} \circ \mathbb{C}$) topology if, and only if, it is closed in the Euclidean topology.*

Proof. We start by proving the claim for subsets of $[0, 1]$. If S is closed in the Zariski topology, there exists a set of pairs of definable functions θ such that

$$S = \mathbb{V}(\theta) = \bigcap_{(f,g) \in \theta} \{s \in [0, 1] \mid f(s) = g(s)\}.$$

It is then enough to prove that for any $(f, g) \in \theta$ the set $\{s \in [0, 1] \mid f(s) = g(s)\}$ is closed in the Euclidean topology. By Lemma 11.5 both f and g are continuous, so by [21, 1.5.4] S is closed. Conversely, if S is closed in the Euclidean topology then there is a function $f: [0, 1] \rightarrow [0, 1]$ that vanishes exactly on S , because closed sets are zero-sets in metrisable spaces. Hence $S = \mathbb{V}(f, 0)$ is closed in the Zariski topology. Thus the Zariski and the Euclidean topologies coincide on $[0, 1]$.

Since $[0, 1]$ is Hausdorff, and since by Lemma 11.5 all definable functions are continuous, by Lemma 8.3 the product and the Zariski topologies coincide, and the proof is complete. \square

Lemma 11.7. *A topological space is compact and Hausdorff if, and only if, it is homeomorphic to a closed subset of $[0, 1]^\mu$ for some μ .*

Proof. This is a standard fact; see e.g. Kelly's embedding lemma [27, Lemma 4.5, pag. 116]. \square

Corollary 11.8. *The variety \mathbf{V}^* is dually equivalent to \mathbf{KHaus} .*

12. CONCLUSIONS

The categorical and the algebraic frameworks presented above are general enough to encompass several dualities in mathematics. The algebraic framework of Part 3, for example, accommodates such standard theories as Priestley duality for distributive lattices [43], Baker-Baynon duality for Riesz spaces and lattice-ordered Abelian groups [6], or Pontryagin duality for compact Abelian groups [41]. Also,

we remark that the dualities for semisimple and finitely presented MV-algebras developed in [34, 35] arose by applying the constructions of the present paper to that specific setting, and thus motivated the present general development. We conclude the paper with a few remarks on further research.

Remark 12.1 (Galois theory of field extensions). Let K be a field, L a fixed extension of K , and let $\text{Gal}_K(L)$ be the group of automorphisms of L that fix K (i.e. if $h \in \text{Gal}_K(L)$ and $k \in K$ then $h(k) = k$).

The classical Galois connection between the intermediate field extensions $K \subseteq F \subseteq L$ and the subgroups of $\text{Aut}_K(L)$ can be recovered as a restriction of the adjunction of Theorem 3.8. To this end we set:

- $\mathbf{T} = \text{Gal}_K(L)$ (i.e., the category with a single object Δ and with arrows the elements of the group $\text{Gal}_K(L)$, composition between them being given by the group operation),
- Δ is the unique object of \mathbf{T} ,
- \mathbf{S} has field extensions of K as objects and elements of $\text{Gal}_K(L)$ as arrows,
- \mathcal{J} is the functor picking the object L of \mathbf{S} and acting identically on arrows.

In this set up the objects of the category \mathbf{R} are pairs (Δ, R) , where $R \subseteq \text{hom}_{\mathbf{T}}^2(\Delta, \Delta)$. As the first component of the pairs can only be Δ , we only write R for an object of \mathbf{R} .

Further, as automorphisms always have an inverse, the condition that p and q act equally on some field F is equivalent to the condition that the automorphism pq^{-1} acts identically on F . We can therefore conceive of relations on $\text{hom}(\Delta, \Delta)$ as subsets of $\text{hom}(\Delta, \Delta)$. The objects of the category \mathbf{D} are pairs (Δ, F) where F is a field such that $K \subseteq F \subseteq L$. For the same reason as above, we only write F for an object of \mathbf{D} .

For any object R in \mathbf{R} , the operator \mathbb{V} specialises to the following:

$$\mathbb{V}(R) = \bigcap \{F \mid \forall h \in R \quad h|_F = \text{id}_F\} \quad (65)$$

For any object F in \mathbf{S} , the operator \mathbb{C} specialises to the following:

$$\mathbb{C}(F) = \{h \in \text{Gal}_K(L) \mid \forall f \in F \quad h(f) = f\} \quad (66)$$

The right-hand set of (65) is often denoted by L^R in classical Galois theory [28, Chapter VI]. The right-hand side of (66) is actually $\text{Aut}_K(F)$.

In a similar way one can also give an account of the Galois connection between fundamental groups and covering spaces of a sufficiently nice topological space; cf. Grothendieck's paper [23].

We have characterised the fixed points of affine adjunction in the algebraic framework through the *Nullstellensatz* and the Stone-Gelfand-Kolmogorov Lemma. The topological side, however, awaits further investigation. In particular, one would like to know when the operator $\mathbb{V} \circ \mathbb{C}$ is topological, and one would like to be able to compare abstractly the $\mathbb{V} \circ \mathbb{C}$ and the product topology on A^μ .

Acknowledgements. The first author thankfully acknowledges partial support from a Research Fellowship at Jesus College, Cambridge, a CARMIN Fellowship at IHÉS-IHP, a Marie Curie INdAM-COFUND-2012 Fellowship and two travel grants of the Dipartimento di Matematica *Federico Enriques* of the Università degli Studi di Milano. The second author gratefully acknowledges partial support by the Italian FIRB “Futuro in Ricerca” grant RBFR10DGUA, which also made possible two

visits of the first author to his Department. The second author is also grateful to the Department of Pure Mathematics and Mathematical Statistics of Cambridge University, to the Category Theory Seminar there, and to the first author, for inviting him to give a talk related to the present paper. The second and third authors further acknowledge partial support from the Italian National Research Project (PRIN2010–11) entitled *Metodi logici per il trattamento dell'informazione*. Parts of this article were written while the second and third authors were kindly hosted by the CONICET in Argentina within the European FP7-IRSES project *MaToMUVI* (GA-2009-247584). The third author gratefully acknowledges partial support by the Marie Curie Intra-European Fellowship for the project “ADAMS” (PIEF-GA-2011-299071).

REFERENCES

- [1] J. Adámek and J. Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.
- [2] J. Adámek, J. Rosický, and E. M. Vitale. *Algebraic theories*, volume 184 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2011. A categorical introduction to general algebra, With a foreword by F. W. Lawvere.
- [3] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
- [4] B. Banaschewski. More on compact Hausdorff spaces and finitary duality. *Canad. J. Math.*, 36(6):1113–1118, 1984.
- [5] M. Barr, J. F. Kennison, and R. Raphael. Isbell duality. *Theory and Applications of Categories*, 20(15):504–542, 2008.
- [6] W. M. Beynon. Duality theorems for finitely generated vector lattices. *Proc. London Math. Soc.* (3), 31:114–128, 1975 part 1.
- [7] G. Birkhoff. Subdirect unions in universal algebra. *Bulletin (New Series) of the American Mathematical Society*, 50(10):764–768, 1944.
- [8] G. Birkhoff. *Lattice theory*, volume 25 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, R.I., third edition, 1979.
- [9] A. Blass and A. Šcedrov. Classifying topoi and finite forcing. *Journal of Pure and Applied Algebra*, 28(2):111–140, 1983.
- [10] S. Burris and H. P. Sankappanavar. *A course in universal algebra*. Graduate texts in Mathematics. Springer-Verlag, 1981.
- [11] O. Caramello. Syntactic characterizations of properties of classifying toposes. *Theory and Applications of Categories*, 26(6):176–193, 2012.
- [12] O. Caramello. Extensions of flat functors and theories of presheaf type. *arXiv:1404.4610 [math.CT]*, 158 pages, 2014.
- [13] O. Caramello. Topos-theoretic background. Online notes, <http://www.oliviacaramello.it/Unification/ToposTheoreticPreliminariesOliviaCaramello.pdf> 2014.
- [14] O. Caramello and N. Wentzlaff. Cyclic theories. *arxiv:math.CT/1406.5479*, 25 pages, 2014.
- [15] D. M. Clark and B. A. Davey. *Natural dualities for the working algebraist*, volume 57. Cambridge University Press, 1998.
- [16] P. M. Cohn. *Universal algebra*, volume 6 of *Mathematics and its Applications*. D. Reidel Publishing Co., Dordrecht-Boston, Mass., second edition, 1981.
- [17] J. B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.
- [18] E. Y. Daniyarova, A. G. Myasnikov, and V. N. Remeslennikov. Algebraic geometry over algebraic structures II: Foundations. *Journal of Mathematical Sciences*, 185(3):389–416, 2012.
- [19] Y. Diers. Affine algebraic sets relative to an algebraic theory. *Journal of Geometry*, 65(1-2):54–76, 1999.
- [20] Y. Diers. Affine algebraic sets relative to an algebraic theory. *Journal of Geometry*, 65(1-2):54–76, 1999.

- [21] R. Engelking. *General topology*, volume 6 of *Sigma Series in Pure Mathematics*. Heldermann Verlag, Berlin, second edition, 1989. Translated from the Polish by the author.
- [22] P. Gabriel and F. Ulmer. *Lokal präsentierbare Kategorien*. Lecture Notes in Mathematics, Vol. 221. Springer-Verlag, Berlin-New York, 1971.
- [23] A. Grothendieck. *Revêtements étales et groupe fondamental: 1. Séminaire de géométrie algébrique du Bois Marie 1960/61*. Springer-Verlag, 1971.
- [24] R. Hartshorne. *Algebraic geometry*. Springer, 1977.
- [25] J. Isbell. Generating the algebraic theory of $C(X)$. *Algebra Universalis*, 15(2):153–155, 1982.
- [26] N. Jacobson. *Basic algebra. II*. W. H. Freeman and Co., San Francisco, Calif., 1980.
- [27] J. L. Kelley. *General topology. 1955*, volume 27 of *Graduate Texts in Mathematics*. Springer-Verlag, 1955.
- [28] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.
- [29] F. W. Lawvere. Functorial semantics of algebraic theories. *Proc. Nat. Acad. Sci. U.S.A.*, 50:869–872, 1963.
- [30] F. W. Lawvere. Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories. *Repr. Theory Appl. Categ.*, 5:1–121, 2004. Reprinted from *Proc. Nat. Acad. Sci. U.S.A.* **50** (1963), 869–872 [MR0158921] and *Reports of the Midwest Category Seminar. II*, 41–61, Springer, Berlin, 1968 [MR0231882].
- [31] F. E. J. Linton. Some aspects of equational categories. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 84–94. Springer, New York, 1966.
- [32] E. G. Manes. *Algebraic theories*. Springer-Verlag, New York-Heidelberg, 1976. Graduate Texts in Mathematics, No. 26.
- [33] V. Marra and L. Reggio. Stone duality above dimension zero: axiomatising the theory of $C(X)$. Manuscript, 2015.
- [34] V. Marra and L. Spada. The dual adjunction between MV-algebras and Tychonoff spaces. *Studia Logica (Special issue dedicated to the memory of Leo Esakia)*, 100(1-2):253–278, 2012.
- [35] V. Marra and L. Spada. Duality, projectivity, and unification in Łukasiewicz logic and MV-algebras. *Ann. Pure Appl. Logic*, 164(3):192–210, 2013.
- [36] R. McKenzie, G. McNulty., and W. Taylor. *Algebras, Lattices, Varieties*. Wadsworth and Brooks/Cole, Monterey CA, 1987.
- [37] J. W. Negreponitis. Duality in analysis from the point of view of triples. *J. Algebra*, 19:228–253, 1971.
- [38] E. Noether. Idealtheorie in Ringbereichen. *Math. Ann.*, 83(1-2):24–66, 1921.
- [39] B. Pareigis. *Categories and functors*. Translated from the German. Pure and Applied Mathematics, Vol. 39. Academic Press, New York-London, 1970.
- [40] J. W. Pelletier and J. Rosický. On the equational theory of C^* -algebras. *Algebra Universalis*, 30(2):275–284, 1993.
- [41] L. Pontryagin. The theory of topological commutative groups. *Annals of Mathematics*, pages 361–388, 1934.
- [42] H.-E. Porst and W. Tholen. Concrete dualities. In *Category theory at work (Bremen, 1990)*, volume 18 of *Res. Exp. Math.*, pages 111–136. Heldermann, Berlin, 1991.
- [43] H. A. Priestley. Ordered sets and duality for distributive lattices. *North-Holland Mathematics Studies*, pages 39–60, 1984.
- [44] J. Ślomiński. The theory of abstract algebras with infinitary operations. *Rozprawy Mat.*, 18:67 pp. (1959), 1959.
- [45] W. Tholen. Nullstellensatz and subdirect representation. *Applied Categorical Structures*, pages 1–23, 2013.
- [46] M. D. Weir. *Hewitt-Nachbin Spaces*, volume 17 of *Mathematics Studies*. Elsevier, 1975.

(O. Caramello) UFR DE MATHÉMATIQUES, UNIVERSITÉ DE PARIS VII, BÂTIMENT SOPHIE GERMAIN, 5 RUE THOMAS MANN, 75205 PARIS CEDEX 13, FRANCE.

(V. Marra) DIPARTIMENTO DI MATEMATICA *Federigo Enriques*, UNIVERSITÀ DEGLI STUDI DI MILANO, VIA CESARE SALDINI 50, 20133 MILANO, ITALY.

(L. Spada) ILLC, UNIVERSITEIT VAN AMSTERDAM, SCIENCE PARK 107, 1098XG AMSTERDAM, THE NETHERLANDS. *Temporary address*.

(L. Spada) DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI SALERNO, VIA GIOVANNI PAOLO II 132, 84084 FISCIANO (SA), ITALY. *Permanent address, on leave.*

E-mail address, V. Marra: `vincenzo.marra@unimi.it`

E-mail address, L. Spada: `lspada@unisa.it`

E-mail address, O. Caramello: `olivia@oliviacaramello.com`